

Non-Abelian topological phases in three spatial dimensions from coupled wires

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Starting from an array of interacting fermionic quantum wires, we construct a family of non-Abelian topologically ordered states of matter in three spatial dimensions (3D). These states of matter inherit their non-Abelian topological properties from the $su(2)_k$ conformal field theories that characterize the constituent interacting quantum wires in the decoupled limit. Thus, the resulting topological phases can be viewed as 3D generalizations of the (bosonic) $su(2)_k$ Read-Rezayi sequence of fractional quantum Hall states. Focusing in detail on the $su(2)_2$ case, we first review how to determine the nature of the non-Abelian topological order (in particular, the topological degeneracy on the torus) in the two-dimensional (2D) case, before generalizing this approach to the 3D case. We also investigate the 2D boundary of the 3D phases, and show for the $su(2)_2$ case that there are anomalous gapless surface states protected by an analog of time-reversal symmetry, similar to the massless Dirac surface states of the noninteracting 3D topological insulator.

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I. INTRODUCTION

A. Motivation

In recent decades, topological order has emerged as a novel paradigm for describing states of matter. Motivated by the study of the fractional quantum Hall effect and chiral spin liquids, theoretical investigations uncovered a rich landscape of topologically ordered phases in two spatial dimensions. The unifying features common to all phases in this landscape are 1) the degeneracy of the ground state when the system is defined

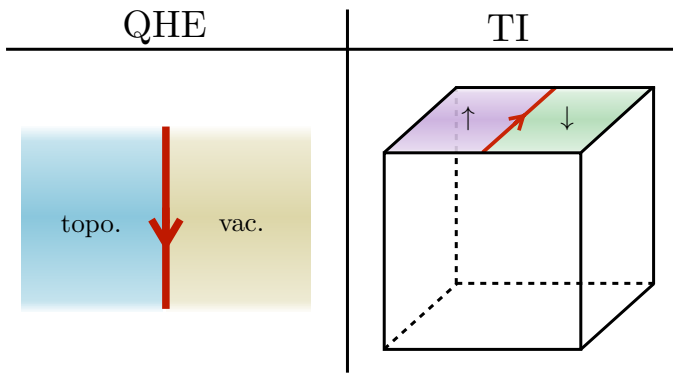


FIG. 1. (Color online) Analogy between the quantum Hall effect (QHE) in (2+1)-dimensional spacetime and the \mathbb{Z}_2 topological insulator (TI) in (3+1)-dimensional spacetime. In the quantum-Hall setting (left), the boundary between the topological phase (blue) and the vacuum (beige) hosts an edge state that realizes one chiral sector of a conformal field theory (red). In the TI setting (right), a domain wall separating two regions with opposite time-reversal breaking fields (labeled “↑” and “↓”) hosts the same chiral mode that appeared on the edge of the quantum Hall system.

on a manifold with nonzero genus [1], and 2) the (intimately related) existence of fractionalized excitations in the gapped bulk [2]. The theoretical understanding of these topologically ordered phases has been placed on a firm mathematical footing rooted in the apparatus of modular tensor categories [3–7]. While numerous problems remain open to investigation, such as the inclusion of symmetries [8–11] and the description of topological phases with fundamental fermions [12–18], this mathematical framework provides an indispensable point of reference in the ongoing effort to understand strongly interacting topological states of matter in two spatial dimensions.

The theoretical proposal [19, 20] and experimental discovery [21–24] of three-dimensional topological insulators protected by time-reversal symmetry (TRS) underscores the natural question of whether a similar understanding of topological order in three spatial dimensions could be achieved. Numerous examples of topologically ordered phases in three spatial dimensions are known, of which discrete gauge theories and their twisted counterparts are perhaps the most elementary [25–28]. There also exists a procedure, the Crane-Yetter/Walker-Wang construction [29–32], that can be used to build certain topological phases in three spatial dimensions. Despite this progress, the question of what kinds of strongly interacting topological phases can exist in (3+1)-dimensional spacetime (3D) is far from settled. This is especially true of non-Abelian topological orders. Furthermore, it is (for the most part) unclear how such topological phases emerge as low-energy descriptions of condensed matter systems, which are conventionally made of electrons and spins that interact in decidedly non-exotic ways.

In this paper, we propose a family of non-Abelian topo-

logical phases in (3+1)-dimensional spacetime. This family of phases can be viewed as completing the following series of analogies between topological phases in two- and three-dimensional space. We begin with the integer quantum Hall effect (IQHE) [33]. This is a topological phase in (2+1)-dimensional spacetime (2D) whose electromagnetic response is encoded by a $U(1)$ Chern-Simons effective action at level 1 [34–36]. It features a unique ground state on the torus, and, on a manifold with boundary, has gapless chiral Dirac fermion edge states [37] described by the affine Lie algebra $u(1)_1$ (see Fig. 1). The noninteracting \mathbb{Z}_2 topological insulator (TI) in 3D can be viewed as inheriting many of its defining properties from the IQHE. For example, although the noninteracting TI respects time-reversal symmetry (TRS) while the IQHE does not, the gapped surface states that emerge when TRS is broken on the surface of the TI feature a Hall response that is exactly half of what is expected in the IQHE case [38]. Thus, a “magnetic domain wall” that separates regions with opposite TRS-breaking fields on the TI surface binds the same gapless chiral Dirac fermion mode that constitutes the edge state of the IQH system (see Fig. 1). This is a direct consequence of the axion electromagnetic response (signaled by a θ term in the effective action) in 3D that characterizes the bulk of the TI [39–41]. When TRS is preserved, the noninteracting \mathbb{Z}_2 TI features a single massless Dirac fermion on its surface. In pure 2D, the existence of a TRS theory of a single massless Dirac fermion is forbidden by the fermion doubling theorem [42]. However, on the surface of a \mathbb{Z}_2 TI, its presence is necessary to ensure that the TRS-breaking surface contribution of the θ term does not spoil TRS on the surface. If TRS is broken on the surface, then a mass term for the Dirac fermion is symmetry-allowed, and the aforementioned surface quantum Hall effect develops. In this sense, the Dirac fermion surface states of the \mathbb{Z}_2 TI are anomalous, and their gaplessness is protected by TRS.

Recent work on so-called fractional TIs (FTIs) in 3D has borne out this analogy to the interacting setting. Indeed, these FTIs can be *defined* as systems in (3+1)-dimensional spacetime with TRS whose bulk axion electromagnetic response is characterized by axion angles θ that are rational multiples of π . Consistency with TRS then demands the presence of topological order in the bulk [43, 44]. In the case $\theta = \pi/k$ with $k \in \mathbb{Z}$, one finds that breaking TRS on the surface of an FTI yields a gapped surface state with Hall conductivity $\sigma_{xy} = (1/2k)e^2/h$. Consequently, a magnetic domain wall on the surface binds a chiral Luttinger-liquid mode described by the affine Lie algebra $u(1)_k$, which is precisely the edge state of the $\nu = 1/k$ Laughlin state in the fractional quantum Hall effect (FQHE) [45]. Moreover, preserving TRS on the surface necessitates the presence of fractionalized gapless excitations on the surface [44]. Hence, FTIs feature fractionalized analogues of the anomalous gapless surface states of the noninteracting \mathbb{Z}_2 TI.

Given the two preceding analogies between the (F)QHE in 2D and the (F)TI in 3D, the following natural question arises. Are there TI-like analogues in 3D of the 2D non-Abelian quantum Hall states? One can focus, for example, on asking this question for the (bosonic) Read-Rezayi quantum Hall sequence in (2+1)-dimensional spacetime [46, 47]. These non-Abelian topological phases are described by an $SU(2)$ Chern-Simons term at level k , and feature chiral edge states described by the affine Lie algebra $su(2)_k$ [45, 48]. Hence, the analogous topological phase in (3+1)-dimensional spacetime would need to satisfy the following three properties. First, it should be time-reversal invariant in the bulk. Second, it should be topologically ordered, in the sense that the ground state manifold on the three-torus must have dimension greater than one. Third, a domain wall between regions on the surface in which TRS is broken in opposite ways should bind a chiral $su(2)_k$ mode. Fourth, there should be gapless surface states protected by TRS. Is it possible to construct such a topological phase? If so, what is the nature of the bulk topological order? These are the questions we address in this paper.

We address the question of the existence of 3D analogues of the $su(2)_k$ fractional quantum Hall states in 2D by building them from scratch. In particular, we employ a coupled-wire construction based on non-Abelian current algebras to construct a topological phase with the desired properties. In this approach, the topological phase in (3+1)-dimensional spacetime is constructed by coupling together many sub-systems, each of which lives in (1+1)-dimensional spacetime (1D), with appropriate many-body interactions. Coupled-wire constructions have been used to construct a variety of strongly-correlated phases in 2D, including non-Fermi liquids [49–51] as well as Abelian and non-Abelian quantum Hall states [52–59]. Moreover, this approach has recently been generalized to 3D, yielding a variety of phases including Weyl semimetals [60, 61], fractional topological insulators [62], and strongly-correlated phases described by emergent Abelian gauge theories [63]. The utility of this approach lies in the fact that numerous analytic techniques exist for quantum field theories in (1+1)-dimensional spacetime, enabling the description of a wide variety of strongly interacting states of matter in a controlled manner. We also argue by example in this work that the coupled-wire approach can be used as a means to search for and characterize new topological phases of matter, like the family of $su(2)_k$ topological phases in 3D constructed here.

B. Outline and summary of results

We now provide an overview of the organization of the paper and summarize the results.

In Sec. II, we review how to bosonize a multi-flavor fermionic wire in terms of the currents associated with the non-Abelian internal symmetry group of the wire [64].

This bosonization scheme has been used to address a wide variety of physical problems in 1D, including the multi-channel Kondo effect [65–67] and marginally-perturbed conformal field theories (CFTs) [68]. In Ref. [69] it was also used as a starting point for the construction of a series of non-Abelian topological phases in 2D. In Sec. IIB, we show how to add intra-wire interactions to drive the fermionic wire to a strong-coupling fixed point described by an $su(2)_k$ CFT. This treatment is crucial for what follows, as these CFTs are used as building blocks for the coupled-wire constructions of the subsequent sections; the non-Abelian topologically ordered phases in 2D and 3D that we construct later in the paper inherit their non-Abelian character from the $su(2)_k$ CFTs.

Next, in Sec. III, we describe how to construct non-Abelian topological phases of matter in 2D starting from a one-dimensional array of decoupled $su(2)_k$ CFTs. This section serves as a prelude to Sec. IV, where the 3D topological phase is constructed. While the $su(2)_k$ topological phases constructed in Sec. III are not new, this section serves two important purposes, on which we now elaborate.

First, Sec. III establishes the approach we later take to construct the 3D topological phases of the following section. This approach can be described as follows. We use $su(2)_k$ current-current interactions to couple channels in neighboring wires that have opposite chirality. These couplings can be viewed as a particular set of interactions between the spin sectors of neighboring wires (see, e.g., [70]). Since these interactions are marginally relevant, they flow to strong coupling and gap the bulk of the array of coupled wires, leaving chiral $su(2)_k$ modes on the boundaries when the model is defined on a cylinder. Once we have shown how to gap the bulk of the array, we move on to characterize the bulk topological order within the coupled-wire construction, focusing on the specific example of $su(2)_2$. (In the quantum Hall parlance, this topological phase is related to the Moore-Read state for bosons at filling factor $\nu = 1$.) The procedure for doing so hinges on using the primary operators of the unperturbed CFTs in each wire to construct nonlocal “string operators” that commute with the interaction term and satisfy a nontrivial algebra among themselves. These string operators can then be used to determine the topological ground-state degeneracy of the coupled-wire theory on the torus. More specifically, when the interwire interactions are turned on and allowed to flow to infinity, these string operators can be used to construct a representation of the ground-state manifold of the resulting strong-coupling fixed point.

Second, the calculation of the ground-state degeneracy of the $su(2)_2$ topological phase in 2D constructed in Sec. III serves to clarify precisely what is meant by a “non-Abelian” topological phase in the context of this paper. (Moreover, such a calculation was not presented in previous coupled-wire constructions of this topological phase, see e.g., Refs. [57] and [69].) In particular, we will show that the algebra of the nonlocal string operators men-

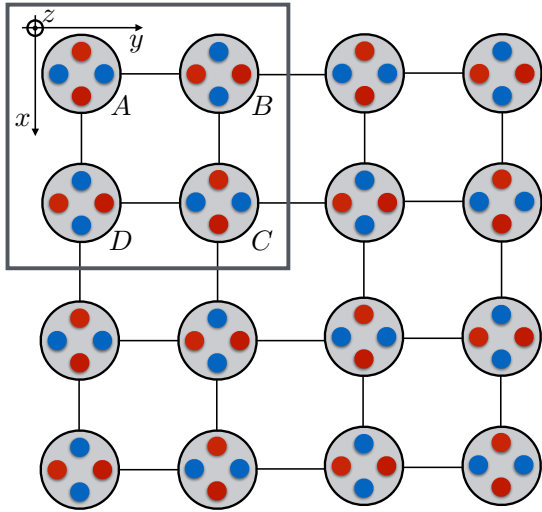


FIG. 2. (Color online) Two-dimensional cross-section of the array of quantum wires used to construct the family of non-Abelian topological phases in (3+1)-dimensional spacetime studied in this work. Each grey disc centered on a site of the square lattice Λ represents a quantum wire aligned along the z -direction of the ambient three-dimensional space with Cartesian coordinates x , y , and z . The colored discs within each grey disc represent normal modes, dispersing along the z -direction, that correspond to different chiral sectors of the $su(2)_k$ conformal field theory. The binary color code distinguishes between left- and right-handed sectors. In this example, each wire contains two right-moving and two left-moving normal modes.

tioned previously suggests the algebra of Wilson loops in a \mathbb{Z}_2 gauge theory; namely, there are four nonlocal string operators that break into two sets of anticommuting operators. Naïve intuition derived from Abelian gauge theory then suggests that the ground-state degeneracy on the torus should be fourfold. However, one finds that one of these four putative ground states cannot reside in the ground-state manifold. The reason for this has deep connections to the non-Abelian algebra of primary operators in the CFT [5], and has come up before in less microscopic studies of related topological phases [71]. In this way, we conclude that the topological degeneracy of the $su(2)_2$ topological phase in 2D is three, rather than four. This exclusion of states from the ground-state manifold based on non-Abelian operator algebras is at the heart of what distinguishes non-Abelian topological phases from Abelian ones, and appears again with a vengeance in the (3+1)-dimensional case.

In Sec. IV, we generalize the results of Sec. III to (3+1)-dimensional spacetime. In this case, the wire construction begins with a two-dimensional array of parallel interacting quantum wires that realize $su(2)_k$ CFTs at low energies (see Fig. 2). We focus again in this section on the case of $su(2)_2$. Most of the important aspects of Sec. III carry through to Sec. IV, albeit with a few important modifications.

First, since we are after some kind of generalization

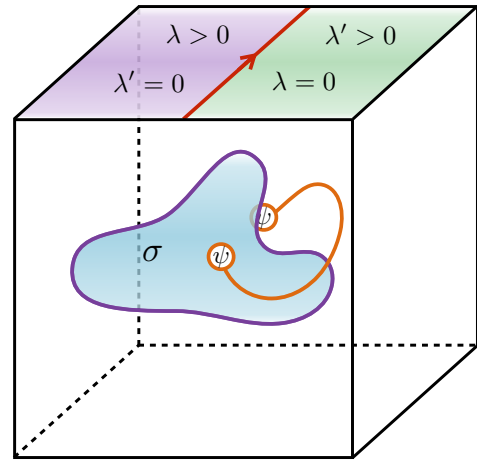


FIG. 3. (Color online) Visual summary of results for the $su(2)_2$ topological phase in (3+1)-dimensional spacetime constructed in this paper. The blue membrane and orange string in the bulk represent membrane and string operators constructed in Sec. IV C. These membrane and string operators are related to the primary fields σ and ψ in the Ising conformal field theory. The boundary of the σ membrane is a stringlike excitation, while the boundaries of the ψ string represent pointlike excitations. The surface features a “magnetic” domain wall separating two regions in which the analogue of time-reversal symmetry (TRS) is broken in two different ways (shown here as a purple and a green region of the surface). The domain wall binds a chiral $su(2)_2$ mode (red).

of a TI, we need some symmetry that plays the role of TRS. In this case, the relevant symmetry is a cousin of TRS: while the individual interwire couplings do not respect TRS, they do so up to a translation by one half of either of the lattice vectors that define the two-dimensional array. [Note that the 2D $su(2)_k$ topological phases constructed in Sec. III break the one-dimensional analogue of this symmetry, just like the associated fractional quantum Hall states break TRS.] This “antiferromagnetic” realization of time-reversal symmetry has appeared in other coupled-wire constructions of TRS topological phases [72, 73]; while it is in some sense an artifact of the wire construction, it does play the same role that TRS plays in the TI (see also Refs. [74, 75]). In particular, it protects anomalous gapless surface states, as we will discuss below.

The second important modification involved in generalizing the 2D results of Sec. III to the 3D setting of Sec. IV is that one must construct nonlocal “membrane” operators” (see Fig. 3), in addition to the string operators of the two-dimensional case, in order to characterize the resulting topological order. These membrane operators can be viewed as the two-dimensional worldsheets of deconfined stringlike excitations, while the string operators can be viewed as the one-dimensional worldlines of deconfined pointlike excitations. The existence of stringlike excitations is crucial for topological order in 3D, since deconfined point particles in three-dimensional space must

have trivial braiding [76]. (Indeed, this is the case for the present topological phase: all of the string operators will be shown to commute.) The non-Abelian algebra of string operators in the 2D example generalizes to a non-Abelian algebra of string and membrane operators 3D. Moreover, there is also a non-Abelian algebra among the membranes. Carrying through the naive Abelian gauge theory counting for the (3+1)-dimensional case and removing states that are excluded based on the non-Abelian operator algebra, we find that the $su(2)_2$ topological phase in (3+1)-dimensional spacetime exhibits a 20-fold topological degeneracy on the three-torus.

Finally, we investigate in Sec. V the surface states of the 3D topological phase constructed in Sec. IV. It is readily seen that, when open boundary conditions are imposed in one of the two directions of the array of wires, there are gapless $su(2)_k$ modes left on the exposed two-dimensional surfaces. The goal of Sec. V is to better understand the fate of these “dangling” modes when they are coupled by marginally relevant local interactions. When these interactions break the TRS analogue, it is straightforward to see that a gapped surface phase results. It is similarly straightforward to see that a “magnetic” domain wall (i.e., a domain wall between two regions of the surface in which the TRS analogue is broken in different ways) binds a chiral $su(2)_k$ current (see Fig. 3).

To probe the phase diagram of the surface when the TRS analogue is preserved, we perform a one-loop renormalization group (RG) analysis of the surface. We find that all couplings on the surface can flow to strong coupling simultaneously, even though neighboring couplings do not commute in general. Furthermore, we allow for surface couplings that break the $SU(2)$ spin-rotation symmetry of the bulk, and find that the surface couplings nevertheless flow towards an $SU(2)$ -symmetric strong-coupling fixed point in a large region of parameter space. Next, we investigate the nature of the $SU(2)$ -symmetric strong-coupling fixed point by an explicit self-consistent mean-field calculation for the $su(2)_2$ case. We find that the surface states are indeed gapless when the TRS analogue is imposed, and that they do not break the symmetry spontaneously (at least at the level of mean field theory). While the question of whether or not there exist gapped, symmetry-preserving surface states of these topological phases, which would then likely exhibit surface topological order [72, 77–83], is interesting, we do not pursue it in this work. However, some investigation along these lines has been carried out in Ref. [73].

In summary, we are able to demonstrate in this paper that it is possible to construct (3+1)-dimensional analogues of the $su(2)_k$ non-Abelian (bosonic) quantum Hall states. For the special case of $su(2)_2$, we characterize the non-Abelian topological order in detail and by explicit calculation, in addition to characterizing the associated surface states. Thus, we arrive at the surprising result that non-Abelian topological states of matter based on conformal field theory can be constructed in

(3+1)-dimensional spacetime.

II. NON-ABELIAN BOSONIZATION OF A SINGLE WIRE

A. Free-fermion wire

Consider a one-dimensional wire containing N_c “colors” of spinful fermions. Its action $S_{0,\text{wire}}$ is the integral over time t and the coordinate z along the wire of the Lagrangian density

$$\mathcal{L}_{0,\text{wire}} := 2 \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha=1}^{N_c} \left(\chi_{L,\sigma,\alpha}^* i \partial_L \chi_{L,\sigma,\alpha} + \chi_{R,\sigma,\alpha}^* i \partial_R \chi_{R,\sigma,\alpha} \right). \quad (2.1)$$

The derivatives $\partial_M \equiv \partial_{z_M}$ ($M = L, R$) are taken with respect to the chiral (light-cone) coordinates

$$z_L \equiv t + z, \quad z_R \equiv t - z. \quad (2.2)$$

We assume periodic boundary conditions along the wire, i.e., in the z -direction. The four Grassmann-valued fields $\chi_{R,\sigma,\alpha}^*$, $\chi_{R,\sigma,\alpha}$, $\chi_{L,\sigma,\alpha}^*$, $\chi_{L,\sigma,\alpha}$ are independent of each other.

Such a wire has an internal symmetry $U(2N_c)_L \times U(2N_c)_R$. The central idea of the series of coupled-wire constructions presented in this paper is to decompose the Lie algebra associated with this symmetry using the following identity (or “conformal embedding”) [84],

$$u(2N_c)_1 = u(1) \oplus su(2)_{N_c} \oplus su(N_c)_2, \quad (2.3)$$

where we have employed the notation g_k for the affine Lie algebra at level k associated with the connected, compact, and simple Lie group G . (For a review of affine Lie algebras, see, e.g., Ref. [84].) Equation (2.3) tells us that the theory (2.1) has three conserved currents — j_R , J_R^a , and J_R^3 — corresponding to the affine Lie algebras $u(1)$, $su(2)_{N_c}$, and $su(N_c)_2$, respectively. (Note that, of course, there are analogous conserved currents j_L , J_L^a , and J_L^3 for the left-handed sector.) We use indices $a = 1, 2, 3$ to label the generators of $SU(2)$ and $a = 1, \dots, N_c^2 - 1$ to label the generators of $SU(N_c)$. In terms of the complex fermions, these currents are given by

$$j_M := \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha=1}^{N_c} \chi_{M,\sigma,\alpha}^* \chi_{M,\sigma,\alpha}, \quad (2.4a)$$

$$J_M^a := \frac{1}{2} \sum_{\sigma,\sigma'=\uparrow,\downarrow} \sum_{\alpha=1}^{N_c} \chi_{M,\sigma,\alpha}^* \sigma_{\sigma\sigma'}^a \chi_{M,\sigma',\alpha}, \quad (2.4b)$$

$$J_M^a := \sum_{\sigma=\uparrow,\downarrow} \sum_{\alpha,\alpha'=1}^{N_c} \chi_{M,\sigma,\alpha}^* T_{\alpha\alpha'}^a \chi_{M,\sigma,\alpha'}, \quad (2.4c)$$

with $M = L, R$. The $U(1)$ currents j_M with $M = L, R$ are associated with charge conservation. The $SU(2)$ currents J_M^a with $M = L, R$ and $a = 1, 2, 3$ are associated with the spin-rotation symmetry. The $SU(N_c)$ currents J_M^a with $M = L, R$ and $a = 1, \dots, N_c^2 - 1$ are associated with the color isospin-rotation symmetry. The generators $\sigma^a/2$ of $SU(2)$ and T^a of $SU(N_c)$ obey the normalizations and the independent algebras

$$\text{tr}(\sigma^a \sigma^b) = 2\delta^{ab}, \quad [\sigma^a, \sigma^b] = 2i\epsilon^{abc} \sigma^c, \quad (2.5a)$$

$$\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad [T^a, T^b] = if^{abc} T^c, \quad (2.5b)$$

where ϵ_{abc} is the Levi-Civita symbol and f_{abc} are the structure constants of $SU(N_c)$. With these definitions, one can build the energy-momentum tensor for the free theory defined by the Lagrangian density (2.1) using the Sugawara construction [65–67, 85] for the energy-momentum tensor in the M -moving sector,

$$T_M[u(2N_c)_1] = T_M[u(1)] + T_M[su(2)_{N_c}] + T_M[su(N_c)_2]. \quad (2.6a)$$

Here,

$$T_M[u(2N_c)_1] := \frac{1}{\pi} \sum_{\sigma=\uparrow, \downarrow} \sum_{\alpha=1}^{N_c} \chi_{M, \sigma, \alpha}^* i \partial_M \chi_{M, \sigma, \alpha}, \quad (2.6b)$$

$$T_M[u(1)] := \frac{1}{4N_c} j_M j_M, \quad (2.6c)$$

$$T_M[su(2)_{N_c}] := \frac{1}{N_c + 2} \sum_{a=1}^3 J_M^a J_M^a, \quad (2.6d)$$

$$T_M[su(N_c)_2] := \frac{1}{2 + N_c} \sum_{a=1}^{N_c^2 - 1} J_M^a J_M^a. \quad (2.6e)$$

With these definitions, it follows that the Hamiltonian density associated with the free Lagrangian density (2.1) is given by

$$\mathcal{H}_{0, \text{wire}} := 2\pi \sum_{M=L, R} \left(T_M[u(1)] + T_M[su(2)_{N_c}] + T_M[su(N_c)_2] \right). \quad (2.7)$$

Rewriting the free theory (2.1) in terms of the currents (2.4) amounts to performing a non-Abelian bosonization of the free theory. This rewriting highlights the fact that a theory of multiple flavors of free fermions can be broken up into independent charge $[u(1)]$, spin $[su(2)_{N_c}]$, and color $[su(N_c)_2]$ sectors.

B. Intra-wire interactions

Having rewritten the free theory (2.1) in terms of the non-Abelian currents (2.4), we now wish to isolate the $su(2)_{N_c}$ spin degrees of freedom by removing the $u(1)$

charge and $su(N_c)_2$ color degrees of freedom from the low-energy sector of the theory. We accomplish this by adding interactions that gap out these degrees of freedom.

To gap out the charge sector, we add to the free Lagrangian density (2.1) the interaction term

$$\mathcal{L}_{\text{int}}[u(1)] := -\lambda_{u(1)} \cos\left(\sqrt{2}(\phi_R + \phi_L)\right). \quad (2.8a)$$

The chiral bosonic fields ϕ_M are defined by the Abelian bosonization identity

$$j_M = -\frac{1}{\sqrt{2}\pi} \partial_M \phi_M. \quad (2.8b)$$

In the fermionic language, the interaction (2.8a) is interpreted as an Umklapp process. It is marginally relevant in the renormalization group (RG) sense, i.e., it flows to strong coupling under RG and gaps the charge sector when $\lambda_{u(1)} > 0$.

To gap out the color sector, we add to the free Lagrangian density (2.1) the interaction term

$$\mathcal{L}_{\text{int}}[su(N_c)_2] := -\lambda_{su(N_c)_2} \sum_{a=1}^{N_c^2 - 1} J_L^a J_R^a, \quad (2.9)$$

where the currents J_M^a are defined in Eqs. (2.4). This current-current interaction is also marginally relevant, flowing to strong coupling for $\lambda_{su(N_c)_2} > 0$.

At the strong-coupling fixed point dominated by the interactions (2.8) and (2.9), the effective Hamiltonian density for the low-energy sector becomes

$$\mathcal{H}_{0, \text{eff}} := 2\pi \left(T_L[su(2)_{N_c}] + T_R[su(2)_{N_c}] \right). \quad (2.10)$$

This is nothing but the Hamiltonian description of the $su(2)_{N_c}$ Wess-Zumino-Witten (WZW) CFT [64, 86] with chiral central charge

$$c[su(2)_{N_c}] = \frac{3N_c}{2 + N_c}. \quad (2.11)$$

Thus, by adding the interactions (2.8) and (2.9) to the free theory (2.1), we can convert a quantum wire containing N_c colors of spinful fermions into a highly nontrivial conformal field theory. The coupled-wire constructions presented in this paper use arrays of these $su(2)_{N_c}$ WZW theories as building blocks for non-Abelian topological phases.

III. WARM-UP: NON-ABELIAN TOPOLOGICAL ORDER IN TWO DIMENSIONS

As a warm-up for Sec. IV, we construct a class of $su(2)_k$ quantum liquids in two spatial dimensions and show, for the case of $k = 2$, how to compute their topological de-

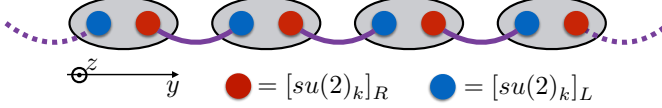


FIG. 4. (Color online) Schematic of the coupled-wire construction for $su(2)_k$ non-Abelian topological orders in two spatial dimensions. Grey ovals represent quantum wires, while red and blue circles represent chiral $su(2)_k$ currents.

generacy on the torus.

A. Definition of the class of models

We begin with a one-dimensional array Λ of parallel nonchiral spinful fermionic quantum wires aligned along the z -direction, each of which is described by the Lagrangian density (2.1) (see Fig. 4). The cardinality of the one-dimensional lattice Λ is

$$|\Lambda| \equiv L_y + 1. \quad (3.1)$$

We set $N_c = k$, where N_c is the number of “colors” of fermions in each wire. Each wire has an internal symmetry $U(2k)_L \times U(2k)_R$, with respect to which we carry out the bosonization procedure of Sec. II. We then gap the $u(1)$ and $su(k)_2$ sectors with the intra-wire interactions discussed in Sec. II A, leaving behind an $su(2)_k$ current algebra for each of the left- and right-moving chiral sectors in every wire. In the Heisenberg picture and in two-dimensional Minkowski space, we denote the chiral $su(2)_k$ currents by $\hat{J}_{M,y}^a(z_M)$ where $M = L, R$ labels the chirality, $a = 1, 2, 3$ labels the $SU(2)$ generators, y labels the wire, and z_M is defined in Eq. (2.2).

To gap out the bulk, we couple nearest-neighbor wires with the $su(2)_k$ current-current interaction (see Fig. 4)

$$\hat{\mathcal{L}}_{bs} \equiv -\hat{\mathcal{H}}_{bs} := - \sum_{y=0}^{L_y - \sigma_{BC}} \sum_{a=1}^3 \lambda^a \hat{J}_{L,y+1}^a \hat{J}_{R,y}^a, \quad (3.2)$$

where $\sigma_{BC} = 0, 1$ for periodic and open boundary conditions, respectively. Like the current-current interaction (2.9), which was used to gap the $su(k)_2$ sector of the theory, the current-current interaction (3.2) is marginally relevant, flowing to strong coupling for $\lambda^a > 0$ and gapping out left- and right-moving gapless degrees of freedom in neighboring wires. When periodic boundary conditions are imposed in the y -direction, i.e., when $\sigma_{BC} = 0$, each chiral current is paired with exactly one

current of the opposite chirality in a neighboring wire, and the full array of quantum wires is gapped. When open boundary conditions are imposed in the y -direction, i.e., when $\sigma_{BC} = 1$, there is a left-moving $su(2)_k$ current at $y = 0$ and a right-moving $su(2)_k$ current at $y = L_y$ that are fully decoupled from the bulk. This edge structure is reminiscent of the $su(2)_k$ non-Abelian Chern-Simons theories [87] and the \mathbb{Z}_k Read-Rezayi quantum Hall states [47].

B. Parafermion representation of the interwire interactions

The nature of the gapped state of matter that results from coupling the array of $su(2)_k$ quantum wires with the interactions (3.2) can be better understood by rewriting the $su(2)_k$ currents in terms of auxiliary degrees of freedom. This rewriting must preserve the $su(2)_k$ current algebra, which is encoded in the operator product expansion (OPE) [84]

$$\hat{J}_{L,y}^a(v) \hat{J}_{L,\tilde{y}}^{\tilde{a}}(w) \sim \delta_{y,\tilde{y}} \left(\frac{(k/2) \delta^{a\tilde{a}}}{v^2 - w^2} + \frac{i \epsilon^{a\tilde{a}b} \hat{J}_{L,y}^b(w)}{v - w} \right), \quad (3.3)$$

for the holomorphic sector $M = L$, and similarly for the antiholomorphic sector $M = R$. (Here, we employ complex coordinates $v \equiv t + iz$, obtained from the chiral coordinate z_L defined in Eq. (2.2) by the analytic continuation $z \rightarrow iz$, and $\bar{v} \equiv t - iz$, obtained from the chiral coordinate z_R also defined in Eq. (2.2) by the same analytic continuation.) The group indices $a, \tilde{a} = 1, 2, 3$, and summation over the repeated index $b = 1, 2, 3$ is implied. The symbol “ \sim ” denotes equality up to nonsingular terms in the limit $v \rightarrow w$.

As shown by Zamolodchikov and Fateev [88] (see Appendix A), the current algebra (3.3) can be represented in terms of \mathbb{Z}_k parafermion and chiral boson operators as follows [84]:

$$\hat{J}_{M,y}^+ =: \sqrt{k} \hat{\Psi}_{M,y} : e^{+i\sqrt{1/k} \hat{\phi}_{M,y}} :, \quad (3.4a)$$

$$\hat{J}_{M,y}^- =: \sqrt{k} : e^{-i\sqrt{1/k} \hat{\phi}_{M,y}} : \hat{\Psi}_{M,y}^\dagger, \quad (3.4b)$$

$$\hat{J}_{M,y}^3 =: i \frac{\sqrt{k}}{2} \partial_M \hat{\phi}_{M,y}, \quad (3.4c)$$

where

$$\hat{J}_{M,y}^\pm =: \hat{J}_{M,y}^1 \pm i \hat{J}_{M,y}^2, \quad (3.4d)$$

and $: \cdot : \hat{H}_{0,\text{eff}}$ denotes normal ordering with respect to the many-body ground state of $\hat{H}_{0,\text{eff}}$ within each wire. Here, the \mathbb{Z}_k parafermions $\hat{\Psi}_{M,y}$ satisfy the equal-time algebra

$$\widehat{\Psi}_{M,y}(t,z)\widehat{\Psi}_{M',y'}(t,z') = \widehat{\Psi}_{M',y'}(t,z')\widehat{\Psi}_{M,y}(t,z) e^{-i\frac{2\pi}{k}\delta_{y,y'}[(-1)^M\delta_{M,M'}\text{sgn}(z-z')+\epsilon_{M,M'}]+i\frac{2\pi}{k}\text{sgn}(y-y')}, \quad (3.4e)$$

$$\widehat{\Psi}_{M,y}^\dagger(t,z)\widehat{\Psi}_{M',y'}^\dagger(t,z') = \widehat{\Psi}_{M',y'}^\dagger(t,z')\widehat{\Psi}_{M,y}^\dagger(t,z) e^{-i\frac{2\pi}{k}\delta_{y,y'}[(-1)^M\delta_{M,M'}\text{sgn}(z-z')+\epsilon_{M,M'}]+i\frac{2\pi}{k}\text{sgn}(y-y')}, \quad (3.4f)$$

$$\widehat{\Psi}_{M,y}(t,z)\widehat{\Psi}_{M',y'}^\dagger(t,z') = \widehat{\Psi}_{M',y'}^\dagger(t,z')\widehat{\Psi}_{M,y}(t,z) e^{+i\frac{2\pi}{k}\delta_{y,y'}[(-1)^M\delta_{M,M'}\text{sgn}(z-z')+\epsilon_{M,M'}]-i\frac{2\pi}{k}\text{sgn}(y-y')}. \quad (3.4g)$$

The sign function above is defined such that $\text{sgn}(0) = 0$. The left- and right-moving labels $M = L, R$ are defined with the convention that $\epsilon_{M,M'}$ is the antisymmetric Levi-Civita symbol obeying $\epsilon_{L,R} = -\epsilon_{R,L} = -1$. Moreover, $(-1)^R = -(-1)^L \equiv 1$. The algebra of the $su(2)_k$ currents holds so long as the equal-time algebra

$$\left[\widehat{\phi}_{M,y}(t,z), \widehat{\phi}_{M',y'}(t,z')\right] = -i2\pi \left[(-1)^M\delta_{y,y'}\delta_{M,M'}\text{sgn}(z-z') + \delta_{y,y'}\epsilon_{M,M'} - \text{sgn}(y-y')\right], \quad (3.4h)$$

is imposed in the chiral bosonic sector. In particular, one verifies that currents defined in different wires commute with one another at equal times when the definitions (3.4) are imposed. Furthermore, one can show that all equal-time commutators between $su(2)_k$ currents differing by their L and R labels also vanish. Finally, the chiral parafermions commute with the chiral bosons at equal times.

This parafermion representation is not unique in two ways. First, as it factorizes a local (observable) operator into the product of two operators, there is an ambiguity with the choice of the phase assigned to each operator-valued factor. The choice for this phase cannot have observable consequences. Second, the dependence on the labels $y \neq y'$ of the equal-time algebra is not unique since many distinct choices accommodate the fact that any two currents belonging to two distinct wires y and y' must always commute. Hence, the dependence on the labels $y \neq y'$ of the parafermion equal-time algebra cannot have observable consequences. We demonstrate that this is true for the case of $su(2)_2$ in Appendix E.

We work with the normalization convention for which the operator $\exp(ia\widehat{\phi}_M)$, for a any real-valued number, has the anomalous scaling dimension a^2 . With this convention, the chiral vertex operator $\exp(i\sqrt{1/k}\widehat{\phi}_M)$, which annihilates a chiral Abelian quasiparticle, has anomalous scaling dimension $1/k$. In turn, the chiral parafermion operator $\widehat{\Psi}_M$ must have the anomalous scaling dimension $1 - (1/k)$, for the current operators have the scaling dimension 1. The expressions (3.4) for the currents are equivalent to the identity [84]

$$su(2)_k \simeq u(1)_k \oplus \mathbb{Z}_k, \quad (3.5a)$$

where

$$\mathbb{Z}_k \equiv \frac{su(2)_k}{u(1)_k}, \quad (3.5b)$$

which states that an $SU(2)$ WZW theory at level k can be interpreted as a direct product of a chiral boson and a \mathbb{Z}_k parafermion conformal field theory.

With these definitions, the interactions (3.2) take the form

$$\widehat{\mathcal{L}}_{\text{bs}} \equiv -\widehat{\mathcal{H}}_{\text{bs}} = -\lambda \frac{k}{2} \sum_{y=0}^{L_y} \left[\left(\widehat{\Psi}_{L,y} : e^{+i\sqrt{\frac{1}{k}}\widehat{\phi}_{L,y}} :: e^{-i\sqrt{\frac{1}{k}}\widehat{\phi}_{R,y+1}} : \widehat{\Psi}_{R,y+1}^\dagger + \text{H.c.} \right) - \frac{1}{2} \left(\partial_L \widehat{\phi}_{L,y} \right) \left(\partial_R \widehat{\phi}_{R,y+1} \right) \right], \quad (3.6)$$

where we have set $\lambda^a \equiv \lambda$ for $a = 1, 2, 3$ for simplicity. (We employ periodic boundary conditions for the remainder of this section.) Written this way, the current-current interactions (3.2) can be reinterpreted as correlated hoppings of fractionalized degrees of freedom between wires. Indeed, viewing $\widehat{\Psi}_{M,y}^\dagger$ as the creation operator for a parafermion with chirality M in wire y , and viewing the vertex operator $: e^{-i\sqrt{\frac{1}{k}}\widehat{\phi}_{M,y}} :$ as the creation operator for an Abelian quasiparticle, we can interpret

the first term in Eq. (3.6) as allowing parafermions to hop between wires so long as an Abelian quasiparticle hops at the same time. Since the composite of these two fractionalized excitations is a boson, as implied by Eqs. (3.4), this correlated hopping process forbids isolated fractionalized degrees of freedom from hopping between wires.

The second term in Eq. (3.6) can be viewed as a chiral density-density interaction between quasiparticles in different wires, in that it penalizes quasiparticle density

imbalances between wires. Its presence leads to confinement of quasiparticle-quasihole pairs created by Abelian vertex-operator products like

$$: e^{+i\sqrt{\frac{1}{k}}\hat{\phi}_{L,y}} :: e^{-i\sqrt{\frac{1}{k}}\hat{\phi}_{R,y}} :. \quad (3.7)$$

The absence of such a density-density term for the parafermions suggests that one *can* create quasiparticle-quasihole pairs with operator products like

$$\hat{\Psi}_{L,y} \hat{\Psi}_{R,y}^\dagger, \quad (3.8)$$

and that the resulting quasiparticles are deconfined (at least in the strong-coupling limit $\lambda \rightarrow \infty$).

Thus, it is reasonable to expect that the strong-coupling fixed point dominated by the interaction (3.6) features (non-Abelian) fractionalized excitations that inherit certain properties of the primary operators of the \mathbb{Z}_k CFT. Moreover, the presence of such fractionalized

excitations implies that the ground-state manifold of the theory in the limit $\lambda \rightarrow \infty$ features a topological degeneracy that cannot be lifted by local perturbations. In the next section, we prove this assertion for the case $k = 2$. Generalizing this analysis to arbitrary k is straightforward.

C. Case study: $su(2)_2$

To characterize the topological order in this gapped state of matter, we will impose periodic boundary conditions in the y - and z -directions and construct nonlocal *string operators* that commute with the interaction $\hat{\mathcal{H}}_{bs}$ defined by Eq. (3.2) and label the topologically degenerate ground states in the limit $\lambda \rightarrow \infty$. We proceed by working through the example of $k = 2$.

The Lagrangian density in this case is

$$\hat{\mathcal{L}}_{bs} \equiv -\hat{\mathcal{H}}_{bs} := -\lambda \sum_{y=0}^{L_y} \left[\left(e^{+i\sqrt{1/2}(\hat{\phi}_{R,y} - \hat{\phi}_{L,y+1})} \hat{\psi}_{L,y+1} \hat{\psi}_{R,y} + \text{H.c.} \right) - \frac{1}{2} \left(\partial_L \hat{\phi}_{L,y+1} \right) \left(\partial_R \hat{\phi}_{R,y} \right) \right], \quad (3.9)$$

which should be compared with Eq. (3.6). The chiral operators

$$\hat{\psi}_{M,y}(t, z) \equiv \hat{\Psi}_{M,y}(t, z) \equiv \hat{\Psi}_{M,y}^\dagger(t, z) \quad (3.10a)$$

with $M = L, R$ are Majorana fermions (i.e., \mathbb{Z}_2 parafermions). Their equal-time exchange algebra is given by Eq. (3.4e) with $k = 2$. We also impose the normalization

$$\lim_{z' \rightarrow z} \hat{\psi}_{M,y}(t, z) \hat{\psi}_{M,y}(t, z') \equiv \lim_{z' \rightarrow z} \delta(z - z') := \mathcal{N}_\delta, \quad (3.10b)$$

where \mathcal{N}_δ is a constant with dimension $[1/\text{length}]$. The chiral bosons $\hat{\phi}_{M,y}$ obey the equal-time algebra (3.4h), as before. Furthermore, the chiral Majorana fermions and the chiral bosons commute at equal times:

$$\left[\hat{\psi}_{M,y}(t, z), \hat{\phi}_{M',y'}(t, z') \right] = 0. \quad (3.11)$$

Observe that the interaction (3.9) is invariant under the M - and y -resolved \mathbb{Z}_2 gauge transformation

$$\hat{\psi}_{M,y}(t, z) \mapsto e^{i\alpha_{M,y}} \hat{\psi}_{M,y}(t, z), \quad (3.12a)$$

$$\hat{\phi}_{M,y}(t, z) \mapsto \hat{\phi}_{M,y}(t, z) + \sqrt{2} \alpha_{M,y}, \quad (3.12b)$$

where the assignments

$$\alpha_{M,y} \in \{0, \pi\} \quad (3.12c)$$

for all chiralities $M = L, R$ and all wires y define the map

$$\alpha : \{M = L, R\} \times \{y = 0, \dots, L_y\} \rightarrow \{0, \pi\}. \quad (3.12d)$$

This transformation is implemented by the operator

$$\hat{\Gamma}_\alpha(t) \equiv \prod_{M=L,R} \prod_{y=0}^{L_y} \hat{\Gamma}_{\alpha_{M,y}}(t) := \hat{\mathcal{U}}_\alpha(t) \hat{\mathcal{Z}}_\alpha(t), \quad (3.13)$$

where the operator

$$\hat{\mathcal{U}}_\alpha(t) \equiv \prod_{M=L,R} \prod_{y=0}^{L_y} \hat{\mathcal{U}}_{\alpha_{M,y}}(t) := \prod_{M=L,R} \prod_{y=0}^{L_y} \exp \left((-1)^M \frac{i\alpha_{M,y}}{2\pi\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{M,y}(t, z) \right) \quad (3.14)$$

acts only on the chiral boson sector of the theory and

implements the transformation (3.12b), and where the

operator

$$\widehat{\mathcal{Z}}_\alpha(t) = \prod_{M=L,R} \prod_{y=0}^{L_y} \widehat{\mathcal{Z}}_{\alpha_{M,y}}(t) \quad (3.15)$$

acts only on the Ising (i.e., \mathbb{Z}_2) sector and implements the transformation (3.12a). The action of the operator $\widehat{\mathcal{U}}_\alpha(t)$ on the chiral bosons follows from the fact that

$$\begin{aligned} \widehat{\mathcal{U}}_{\alpha_{M,y}}(t) \widehat{\phi}_{M',y'}(t,z) \widehat{\mathcal{U}}_{\alpha_{M,y}}^\dagger(t) &= \widehat{\phi}_{M',y'}(t,z) \\ &\quad + \sqrt{2} \alpha_{M,y} \delta_{y,y'} \delta_{M,M'} \end{aligned} \quad (3.16)$$

holds for any pair of chiralities $M, M' = L, R$, for any pair of wires y, y' , and for any t and z [see Eq. (3.4h)]. The action of the operator $\widehat{\mathcal{Z}}_\alpha(t)$ follows from the definition of $\widehat{\mathcal{Z}}_{\alpha_{M,y}}(t)$ in terms of the fermion parity operator in the wire y , which is somewhat involved and will not be presented here.

1. Twist operators

Each nonchiral wire, with its $[su(2)_2]_L \times [su(2)_2]_R$ current algebra, contains the primary field (the “twist field”) $\widehat{\sigma}_{M,y}$ (c.f. Appendix B) in addition to the primary field $\widehat{\psi}_{M,y}$. For any pair of wires y and y' , we posit the following OPEs (using the complex coordinates $v \equiv t + iz$ and $v' \equiv t' + iz'$):

$$\widehat{\psi}_{L,y}(v) \widehat{\sigma}_{L,y'}(v') = \delta_{y,y'} \frac{C_{\psi\sigma}^\sigma}{(v - v')^{1/2}} \widehat{\sigma}_{L,y}(v) + \dots, \quad (3.17a)$$

$$\widehat{\psi}_{R,y}(\bar{v}) \widehat{\sigma}_{R,y'}(\bar{v}') = \delta_{y,y'} \frac{C_{\psi\sigma}^\sigma}{(\bar{v} - \bar{v}')^{1/2}} \widehat{\sigma}_{R,y}(\bar{v}) + \dots, \quad (3.17b)$$

$$\widehat{\psi}_{L,y}(v) \widehat{\sigma}_{R,y'}(\bar{v}') = \widehat{\psi}_{R,y}(\bar{v}) \widehat{\sigma}_{L,y'}(v') = 0 + \dots, \quad (3.17c)$$

where the structure constants obey the symmetry condition

$$C_{\psi\sigma}^\sigma = C_{\sigma\psi}^\sigma, \quad (3.17d)$$

and \dots stands for nonsingular terms. Determining the equal-time algebra of the twist fields and the Majorana fields requires one to restrict the above OPE to the real line in the complex plane. Because of the symmetry condition (3.17d) on the structure constants, exchanging the order of the fields $\widehat{\psi}_{L,y}(v)$ and $\widehat{\sigma}_{L,y'}(v')$ on the left-hand side of Eqs. (3.17a), say, is equivalent to exchanging v and v' . However, when we restrict the OPE to equal times, information about the handedness of this exchange is lost, see Fig. 5. We therefore adopt the following convention for their equal-time operator algebra in two-dimensional

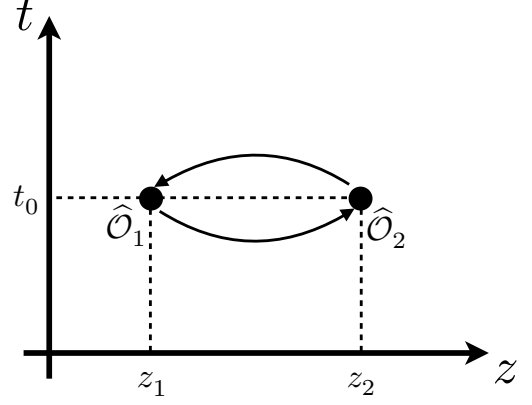


FIG. 5. Counterclockwise monodromy of two operators $\widehat{\mathcal{O}}_1(t_0, z_1)$ and $\widehat{\mathcal{O}}_2(t_0, z_2)$ in the complex plane. When the operators $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_2$ are evaluated at equal times, their exchange can be viewed as monodromy in the complex plane, provided that the handedness of the monodromy is specified. We adopt the convention that the (holomorphic) operator with the larger value of z is passed counterclockwise around the operator with the smaller value of z , resulting in the factors of $\text{sgn}(z - z')$ that appear in the exchange algebras for the primary operators in this section.

Minkowski space. We make the choice

$$\begin{aligned} \widehat{\psi}_{L,y}(t, z) \widehat{\sigma}_{L,y'}(t, z') &= \widehat{\sigma}_{L,y'}(t, z') \widehat{\psi}_{L,y}(t, z) \\ &\quad \times e^{-i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z - z')}, \end{aligned} \quad (3.18a)$$

$$\begin{aligned} \widehat{\psi}_{R,y}(t, z) \widehat{\sigma}_{R,y'}(t, z') &= \widehat{\sigma}_{R,y'}(t, z') \widehat{\psi}_{R,y}(t, z) \\ &\quad \times e^{+i \frac{\pi}{2} \delta_{y,y'} \text{sgn}(z - z')}, \end{aligned} \quad (3.18b)$$

$$\widehat{\psi}_{L,y}(t, z) \widehat{\sigma}_{R,y'}(t, z') = \widehat{\sigma}_{R,y'}(t, z') \widehat{\psi}_{L,y}(t, z), \quad (3.18c)$$

for any pair of wires y and y' and for any $z \neq z'$. The appearance of the phase $\pi/2$ is fixed by the OPE (3.17a) and (3.17b) and the sign $\text{sgn}(z - z')$ is used to keep track of the handedness of the exchange. This choice of sign convention for the phase $\pi/2$ is equivalent to a choice of analytic continuation into the complex plane in order to regularize the equal-time exchange of the two operators.

The equal-time algebra of two twist operators is more subtle. For any pair of wires y and y' , the OPE of two twist fields in the complex plane is given by [c.f. Eq. (B5b)]

$$\begin{aligned} \widehat{\sigma}_{L,y}(v) \widehat{\sigma}_{L,y'}(v') &= \delta_{y,y'} \frac{C_{\sigma\sigma}^1}{(v - v')^{1/8}} \\ &\quad + \delta_{y,y'} C_{\sigma\sigma}^\psi (v - v')^{3/8} \widehat{\psi}_{L,y}(v), \end{aligned} \quad (3.19a)$$

$$\begin{aligned} \widehat{\sigma}_{R,y}(\bar{v}) \widehat{\sigma}_{R,y'}(\bar{v}') &= \delta_{y,y'} \frac{C_{\sigma\sigma}^1}{(\bar{v} - \bar{v}')^{1/8}} \\ &\quad + \delta_{y,y'} C_{\sigma\sigma}^\psi (\bar{v} - \bar{v}')^{3/8} \widehat{\psi}_{R,y}(\bar{v}), \end{aligned} \quad (3.19b)$$

$$\hat{\sigma}_{L,y}(v)\hat{\sigma}_{R,y'}(\bar{v}') = \hat{\sigma}_{R,y}(\bar{v})\hat{\sigma}_{L,y'}(v') = 0 + \dots \quad (3.19c)$$

Since there are two singular terms appearing on the right-hand side of Eqs. (3.19a) and (3.19b), the product of two chiral twist fields must be defined with care. In particular, correlation functions involving an even number of chiral twist fields are not well-defined unless the fusion channel $\mathbb{1}$ or ψ is specified [89]. We choose an equal-time operator algebra that reflects this ambiguity in the definition of chiral correlation functions involving the twist field. Thus, we define the equal-time algebra

$$\begin{aligned} \hat{\sigma}_{L,y}(t, z)\hat{\sigma}_{L,y'}(t, z') &= \hat{\sigma}_{L,y'}(t, z')\hat{\sigma}_{L,y}(t, z) \\ &\times \begin{cases} e^{-i\frac{\pi}{8}\delta_{y,y'}\text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \mathbb{1}, \\ e^{+i\frac{3\pi}{8}\delta_{y,y'}\text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \psi, \end{cases} \end{aligned} \quad (3.20a)$$

$$\begin{aligned} \hat{\sigma}_{R,y}(t, z)\hat{\sigma}_{R,y'}(t, z') &= \hat{\sigma}_{R,y'}(t, z')\hat{\sigma}_{R,y}(t, z) \\ &\times \begin{cases} e^{+i\frac{\pi}{8}\delta_{y,y'}\text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \mathbb{1}, \\ e^{-i\frac{3\pi}{8}\delta_{y,y'}\text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \psi \end{cases} \end{aligned} \quad (3.20b)$$

$$\hat{\sigma}_{L,y}(t, z)\hat{\sigma}_{R,y'}(t, z') = \hat{\sigma}_{R,y'}(t, z')\hat{\sigma}_{L,y}(t, z), \quad (3.20c)$$

in two-dimensional Minkowski space for any pair of wires y and y' and for any $z \neq z'$. We have used the shorthand notation $\sigma \times \sigma = \mathbb{1}$ and $\sigma \times \sigma = \psi$ to distinguish the two possible fusion outcomes. It is important to stress here that this equal-time algebra is not well-defined unless one specifies a fusion channel. This ambiguity is essential; its origin is physical, and it reflects the non-Abelian nature of the twist field. We will see in the next section that this ambiguity has important consequences for the topological degeneracy.

2. String operators and topological degeneracy on the two-torus

We shall consider two distinct wires y and y' and a coordinate z along any one of these wires. Periodic boundary conditions are imposed both along the y -direction and along the z -direction. Hence, the one-dimensional array of wires has the topology of a torus. We are going to construct the equal-time algebra

$$\{\hat{\Gamma}_1^\sigma, \hat{\Gamma}_2^\psi\} = 0 \quad (3.21)$$

for a first pair of nonlocal operators $\hat{\Gamma}_1^\sigma$ and $\hat{\Gamma}_2^\psi$. This pair will be shown to commute with the interaction (3.9). The nonlocal, nonunitary operator $\hat{\Gamma}_1^\sigma$ can be thought of as creating a pair of pointlike “ σ ” excitations, transporting them in opposite directions around a noncontractible cycle of the torus along the y -direction, and then annihilating them. Likewise, the nonlocal operator $\hat{\Gamma}_2^\psi$ can be thought of as implementing a similar process for a pair of pointlike “ ψ ” excitations. Similarly, we are going to

construct the equal-time algebra

$$\{\hat{\Gamma}_1^\psi, \hat{\Gamma}_2^\sigma\} = 0 \quad (3.22)$$

for a second pair of nonlocal operators $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\sigma$. This pair will also be shown to commute with the interaction (3.9), modulo appropriate regularization of the operator $\hat{\Gamma}_2^\sigma$, as we will discuss. The nonlocal, unitary operator $\hat{\Gamma}_1^\psi$ can be thought of as creating a pair of “ ψ ” excitations, transporting them in opposite directions around a noncontractible cycle of the torus along the y -direction, and then annihilating them. The nonlocal, nonunitary operator $\hat{\Gamma}_2^\sigma$ can be thought of as implementing the same process for a pair of “ σ ” excitations. If we denote a ground state of the interaction (3.9) by $|\Omega\rangle$, we shall demonstrate that the three states

$$|\Omega\rangle, \quad |\hat{\Gamma}_1^\sigma\rangle := \hat{\Gamma}_1^\sigma |\Omega\rangle, \quad |\hat{\Gamma}_2^\sigma\rangle := \hat{\Gamma}_2^\sigma |\Omega\rangle, \quad (3.23)$$

are linearly independent ground states of the interaction (3.9). The proof of this claim relies on the vanishing equal-time commutators

$$[\hat{\Gamma}_2^\psi, \hat{\Gamma}_1^\psi] = 0, \quad (3.24)$$

$$[\hat{\Gamma}_1^\sigma, \hat{\Gamma}_1^\psi] = 0, \quad (3.25)$$

and

$$[\hat{\Gamma}_2^\psi, \hat{\Gamma}_2^\sigma] = 0. \quad (3.26)$$

Crucially, however, the exchange algebra of the nonlocal operators $\hat{\Gamma}_1^\sigma$ and $\hat{\Gamma}_2^\sigma$ suffers from the same ambiguity as that found on the right-hand side of Eq. (3.20). This is why one cannot infer from Eqs. (3.21)–(3.26) that the state

$$\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma |\Omega\rangle \quad (3.27)$$

is linearly independent from the states (3.23). (See also Appendix C.)

Proof. The proof consists of three steps.

Step 1: Majorana- (ψ -) string operators The first string operators that we will construct are the Majorana string operators. We begin with strings running along the y -direction, perpendicular to the wires. These strings are built from the local Majorana bilinears

$$\hat{\mathcal{O}}_y^\psi(t, z) := \mathcal{N}_\delta^{-1} \hat{\psi}_{L,y}(t, z) \hat{\psi}_{R,y}(t, z) \quad (3.28a)$$

for any $0 < z < L_z$. The dimensionful constant \mathcal{N}_δ was defined in Eq. (3.10b). With this dimensionful constant included, $\hat{\mathcal{O}}_y^\psi(t, z)$ is a local unitary operator,

$$\hat{\mathcal{O}}_y^\psi(t, z) [\hat{\mathcal{O}}_y^\psi(t, z)]^\dagger = \mathbb{1}. \quad (3.28b)$$

Using Eq. (3.4e) for $k = 2$, we see that a product of Majorana bilinears in neighboring wires commutes with

the part of the interaction (3.9) that connects the two wires, since

$$\hat{\mathcal{O}}_y^\psi(t, z) \hat{\mathcal{O}}_{y+1}^\psi(t, z) \hat{\psi}_{L, y+1}(t, z') \hat{\psi}_{R, y}(t, z') = \hat{\psi}_{L, y+1}(t, z') \hat{\psi}_{R, y}(t, z') \hat{\mathcal{O}}_y^\psi(t, z) \hat{\mathcal{O}}_{y+1}^\psi(t, z), \quad (3.29)$$

and because $\hat{\mathcal{O}}_y^\psi(t, z)$ commutes with any operator from the chiral-boson sector of the theory. Thus, the nonlocal operator

$$\hat{\Gamma}_1^\psi(t, z) := \prod_{y=0}^{L_y} \hat{\mathcal{O}}_y^\psi(t, z) \quad (3.30)$$

commutes with the interaction (3.9) for any value of $0 \leq z < L_z$ when periodic boundary conditions are imposed in the y -direction. The nonlocal operator (3.30) is a member of the family

$$\hat{\Gamma}_1^\psi(t, z_1, \dots, z_{L_y}) := \hat{\psi}_{L, 1}(t, z_1) \hat{\psi}_{R, 1}(t, z_2) \hat{\psi}_{L, 2}(t, z_2) \hat{\psi}_{R, 2}(t, z_3) \cdots \hat{\psi}_{L, L_y}(t, z_{L_y}) \hat{\psi}_{R, L_y}(t, z_1) \quad (3.31)$$

of operators, which all commute with the Hamiltonian defined by Eq. (3.2) for any values of $0 \leq z_1, \dots, z_{L_y} < L_z$ when periodic boundary conditions are imposed in the y -direction. Any Majorana string operator from the family (3.31) can be viewed as creating a pair of ψ excitations and transporting one of them around a noncontractible loop that encircles the torus in the y -direction (a noncontractible cycle along the y -direction), before annihilating it with its partner.

To construct a Majorana string running along the z -direction, parallel to the wires, we consider the bilocal

operator

$$\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) := i \mathcal{N}_\delta^{-1} \hat{\psi}_{L, y}(t, z_1) \hat{\psi}_{R, y+1}(t, z_2), \quad (3.32a)$$

for any $0 \leq z_1, z_2 < L_z$, which obeys

$$\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) \left[\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) \right]^\dagger = \mathbb{1}. \quad (3.32b)$$

We have chosen the multiplicative factor i so that

$$\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) = \left[\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) \right]^\dagger. \quad (3.32c)$$

Hence, $\hat{\mathcal{O}}_y^\psi(t, z_1, z_2)$ is a bilocal Hermitian and unitary operator that also obeys

$$\hat{\mathcal{O}}_y^\psi(t, z_1, z_2) \hat{\psi}_{R, y+1}(t, z) \hat{\psi}_{L, y}(t, z) = \hat{\psi}_{R, y+1}(t, z) \hat{\psi}_{L, y}(t, z) \hat{\mathcal{O}}_y^\psi(t, z_1, z_2), \quad (3.33)$$

as a result of Eq. (3.4e) for $k = 2$. Now define the nonlocal operator

$$\hat{\Gamma}_{2, y}^\psi(t) := \hat{\mathcal{O}}_y^\psi(t, 0, L_z) = \left[\hat{\Gamma}_{2, y}^\psi(t) \right]^\dagger. \quad (3.34)$$

It commutes with the interaction (3.9) if it is understood that the integral in Eq. (3.9) is to be interpreted as

$$\int_0^{L_z} dz (\cdots) = \lim_{\epsilon \rightarrow 0} \int_{0+\epsilon}^{L_z-\epsilon} dz (\cdots). \quad (3.35)$$

This Majorana string operator can be viewed as trans-

porting a Majorana fermion around a noncontractible loop that encircles the torus in the z -direction (a noncontractible cycle along the z -direction).

The equal-time commutation relation between the string operators (3.30) with $0 < z < L_z$ and (3.34) is computed using Eq. (3.4e) for $k = 2$. It is simply the commutative rule

$$\hat{\Gamma}_1^\psi(t, z) \hat{\Gamma}_{2, y}^\psi(t) = \hat{\Gamma}_{2, y}^\psi(t) \hat{\Gamma}_1^\psi(t, z). \quad (3.36)$$

This result reflects the fact that fermions have trivial braiding statistics under a full exchange. We have estab-

lished Eq. (3.24) provided we make the identifications

$$\hat{\Gamma}_1^\psi(t, z) \rightarrow \hat{\Gamma}_1^\psi, \quad \hat{\Gamma}_{2,y}^\psi(t) \rightarrow \hat{\Gamma}_2^\psi. \quad (3.37)$$

Step 2: Twist- (σ -) string operators We next construct string operators associated with the Ising twist field σ . We proceed according to a strategy similar to the one used for the Majorana strings. To construct a twist string along the y -direction, let $0 < z, z' < L_z$ and consider the

$$\hat{\Gamma}_1^\sigma(t, z_1, \dots, z_{L_y}) := \hat{\sigma}_{L,1}(t, z_1) \hat{\sigma}_{R,1}(t, z_2) \hat{\sigma}_{L,2}(t, z_2) \hat{\sigma}_{R,2}(t, z_3) \cdots \hat{\sigma}_{L,L_y}(t, z_{L_y}) \hat{\sigma}_{R,L_y}(t, z_1) \quad (3.40)$$

of operators that commute with the Hamiltonian defined by Eq. (3.2) for any values of $0 < z_1, \dots, z_{L_y} < L_z$ when periodic boundary conditions are imposed in the y -direction. Any σ -string operator from the family (3.40) can be interpreted as creating a pair of σ excitations and transporting one of them around a noncontractible cycle along the y -direction, before annihilating it with its partner.

We first observe that the operators $\hat{\Gamma}_1^\psi(t, z)$ and $\hat{\Gamma}_1^\sigma(t, z')$ commute with one another for any z and z' , as one can show using the equal-time algebra (3.18),

$$\hat{\Gamma}_1^\psi(t, z) \hat{\Gamma}_1^\sigma(t, z') = \hat{\Gamma}_1^\sigma(t, z') \hat{\Gamma}_1^\psi(t, z). \quad (3.41)$$

We have established Eq. (3.25) provided we make the identifications

$$\hat{\Gamma}_1^\psi(t, z) \rightarrow \hat{\Gamma}_1^\psi, \quad \hat{\Gamma}_1^\sigma(t, z') \rightarrow \hat{\Gamma}_1^\sigma. \quad (3.42)$$

We claim that the σ -string $\hat{\Gamma}_1^\sigma$ can be interpreted as an operator that “twists,” from antiperiodic to periodic, the boundary conditions of a Majorana fermion that encircles the torus in the z -direction. To see that this is the case, we use Eqs. (3.18) to show that the equal-time operator algebra

$$\hat{\Gamma}_{2,y}^\psi(t) \hat{\Gamma}_1^\sigma(t, z) = -\hat{\Gamma}_1^\sigma(t, z) \hat{\Gamma}_{2,y}^\psi(t) \quad (3.43)$$

holds. We further recall that the operator $\hat{\Gamma}_{2,y}^\psi(t)$ transports a Majorana fermion around the torus along the z -direction. Thus, Eq. (3.43) shows that the amplitude for transporting a Majorana fermion around the torus and then applying the operator $\hat{\Gamma}_1^\sigma(t, z)$ differs by a minus sign from the amplitude for applying the operator $\hat{\Gamma}_1^\sigma(t, z)$ and then transporting a Majorana fermion around the torus.

local twist-field bilinears

$$\hat{\mathcal{O}}_y^\sigma(t, z) := \hat{\sigma}_{L,y}(t, z) \hat{\sigma}_{R,y}(t, z). \quad (3.38)$$

Using Eq. (3.18), we find that the equal-time product of such bilinears over all wires, namely

$$\hat{\Gamma}_1^\sigma(t, z) := \prod_{y=0}^{L_y} \hat{\mathcal{O}}_y^\sigma(t, z), \quad (3.39)$$

commutes with the interaction (3.9) for any value $0 < z < L_z$ when periodic boundary conditions are imposed in the y -direction. This nonlocal, nonunitary operator is a member of the family

This is precisely the action of an operator that twists the boundary conditions of a Majorana fermion.

In deriving Eq. (3.43), we have established Eq. (3.21) provided that we make the identifications

$$\hat{\Gamma}_{2,y}^\psi(t) \rightarrow \hat{\Gamma}_2^\psi, \quad \hat{\Gamma}_1^\sigma(t, z) \rightarrow \hat{\Gamma}_1^\sigma. \quad (3.44)$$

Next, we seek an operator that twists the boundary conditions of a Majorana fermion encircling the torus along the y -direction. Such an operator is necessarily nonlocal. To construct this operator, we proceed in two steps. First, we observe that the operator product

$$\hat{\sigma}_{M,y'}(t, z_1) \hat{\sigma}_{M,y'}(t, z_2) \quad (3.45)$$

can be used to twist the boundary conditions of a Majorana fermion for an appropriate choice of the points z_1 and z_2 . To see this, we note using Eqs. (3.18) that the equal-time exchange algebras

$$\begin{aligned} & \hat{\psi}_{M,y'}(t, z) \hat{\sigma}_{M,y'}(t, z_1) \hat{\sigma}_{M,y'}(t, z_2) \\ &= \hat{\sigma}_{M,y'}(t, z_1) \hat{\sigma}_{M,y'}(t, z_2) \hat{\psi}_{M,y'}(t, z) \\ & \quad \times e^{i(-1)^M \frac{\pi}{2} \text{sgn}(z-z_1)} e^{i(-1)^M \frac{\pi}{2} \text{sgn}(z-z_2)} \end{aligned} \quad (3.46a)$$

and

$$\begin{aligned} & \hat{\psi}_{M,y}(t, z) \hat{\sigma}_{M,y'}(t, z_1) \hat{\sigma}_{M,y'}(t, z_2) \\ &= \hat{\sigma}_{M,y'}(t, z_1) \hat{\sigma}_{M,y'}(t, z_2) \hat{\psi}_{M,y}(t, z) \end{aligned} \quad (3.46b)$$

hold for any $y \neq y'$. Equation (3.46a) states that the operator defined in Eq. (3.45) anticommutes with the Majorana fermion $\hat{\psi}_{M,y}(t, z)$ for z_1 and z_2 such that $\text{sgn}(z - z_1) = \text{sgn}(z - z_2)$. This is only accomplished for all z if $z_1 = z_2$. However, we cannot set $z_1 = z_2$ identically, since then the operator defined in Eq. (3.45)

is singular. Nevertheless, if we perform point splitting such that $z_2 = \lim_{\epsilon \rightarrow 0} (z_1 + \epsilon)$, then the operator defined in Eq. (3.45) only fails to anticommute with the operator $\hat{\psi}_{M,y'}(t, z)$ over a vanishingly small region of size ϵ . Thus, for any choice of z_1 and $z_2 = \lim_{\epsilon \rightarrow 0} (z_1 + \epsilon)$, the operator defined in Eq. (3.45) can be used to twist the boundary conditions of a Majorana fermion encircling the torus along the y -direction.

Before proceeding with the construction of the second twist-field string operator, let us note that any product of the form (3.45), which contains two chiral twist fields in the *same* wire, is ill-defined unless a fusion channel is specified. By analogy with the construction of the string operator $\hat{\Gamma}_1^\sigma$ defined in Eq. (3.39), we wish to build a string operator that can be interpreted as creating a pair of σ excitations out of the vacuum, dragging one of them around the torus, and then annihilating the pair. The natural choice for the fusion channel in the product (3.45) is to specify that the two $\hat{\sigma}_{M,y'}$ operators therein fuse to the identity operator $\mathbb{1}$. In addition to providing a sensible parallel with the construction of $\hat{\Gamma}_1^\sigma$, this choice agrees with the choice made in the construction of the operator that tunnels an $e/4$ quasiparticle across a quantum point contact in the Moore-Read state [89].

The second step in the construction of the twist-field string operator acting along the z -direction is to rectify the following problem. Although the operator (3.45) anticommutes with a single chiral Majorana operator in wire y' , it can also anticommute with terms in the interaction (3.9) that connect the wire y' to its nearest neighbors. For example, if $0 < \epsilon < |z - z_1|$,

$$\begin{aligned} \hat{\sigma}_{R,y'}(t, z_1) \hat{\sigma}_{R,y'}(t, z_1 + \epsilon) \hat{\psi}_{L,y'+1}(t, z) \hat{\psi}_{R,y'}(t, z) \\ = -\hat{\psi}_{L,y'+1}(t, z) \hat{\psi}_{R,y'}(t, z) \\ \times \hat{\sigma}_{R,y'}(t, z_1) \hat{\sigma}_{R,y'}(t, z_1 + \epsilon). \end{aligned} \quad (3.47)$$

However, we can fix this by applying an operator that acts only on the bosonic sector of the theory, namely the operator $\hat{U}_{\alpha_{R,y'}=\pi}$ defined in Eq. (3.14). Thus, we claim that the nonlocal, nonunitary operator

$$\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon) := \hat{U}_{\alpha_{R,y'}=\pi} \hat{P}_\mathbb{1} \hat{\sigma}_{R,y'}(t, z_1) \hat{\sigma}_{R,y'}(t, z_1 + \epsilon) \hat{P}_\mathbb{1} \quad (3.48)$$

is the operator we seek, since it both twists the boundary conditions for a Majorana fermion encircling the torus along the y -direction and commutes with the interaction (3.2) in the limit $\epsilon \rightarrow 0$.¹

We make three observations about the definition

¹ There is a caveat here that has extremely important implications for the derivation of the topological degeneracy, and that we discuss in detail in Appendix C. The caveat is that, although the operator $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ commutes with the interaction (3.9) in the limit $\epsilon \rightarrow 0$, this need not be (and, in fact, *is not*) true of the op-

(3.48). We first note that the choice $M = R$ is arbitrary, as is the choice of the base point z_1 . Second, to make sense of the product of two $\hat{\sigma}_{R,y'}$ operators belonging to the same wire y' in Eq. (3.48), we introduced the projection operator $\hat{P}_\mathbb{1}$ that projects this product into the fusion channel $\sigma \times \sigma = \mathbb{1}$. One can show that this projector does not affect the algebra of twist operators $\hat{\sigma}_{M,y}$ and Majorana operators $\hat{\psi}_{M,y}$. The necessity of performing such a projection was discussed in the previous paragraph. Third, the operator $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ is a “string operator” in the following sense. It is able to detect, by means of the algebras (3.18) and (3.20), the insertion of a ψ - or σ -operator at any point $0 \leq z < L_z$ such that z does not lie between z_1 and $z_1 + \epsilon$. Hence, in the limit $\epsilon \rightarrow 0$, the operator $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ can detect the insertion of such operators at any point in the wire.

To see that the operator $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ indeed twists the boundary conditions appropriately, we use Eqs. (3.46) with $z_1 = 0$ and $z_2 = 0 + \epsilon$ (with $\epsilon > 0$ infinitesimal) to calculate

$$\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon) \hat{\Gamma}_1^\psi(t, z) = -\hat{\Gamma}_1^\psi(t, z) \hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon) \quad (3.49)$$

for any $1 \leq y' \leq L_y$, since $\hat{\Gamma}_{2,y'}^\sigma(t)$ can only fail to commute with the chiral Majorana operators making up $\hat{\Gamma}_1^\psi(t, z)$ belonging to wire y' . Thus, we have established Eq. (3.22) provided we make the identifications

$$\hat{\Gamma}_1^\psi(t, z) \rightarrow \hat{\Gamma}_1^\psi, \quad \hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon) \rightarrow \hat{\Gamma}_2^\sigma \quad (3.50)$$

for infinitesimal $\epsilon > 0$. (We demand that the hierarchy $0 < \epsilon < |z - z_1|$ holds for any z, z_1 . Hence, when we say that ϵ is “infinitesimal”, we mean it in this sense.)

By assumption $y \neq y'$. Hence, the operators $\hat{\Gamma}_{2,y}^\psi \rightarrow \hat{\Gamma}_2^\psi$ and $\hat{\Gamma}_{2,y'}^\sigma \rightarrow \hat{\Gamma}_2^\sigma$ commute with one another according to Eq. (3.46b), i.e.,

$$\hat{\Gamma}_2^\sigma \hat{\Gamma}_2^\psi = \hat{\Gamma}_2^\psi \hat{\Gamma}_2^\sigma. \quad (3.51)$$

We have established Eq. (3.26).

Step 3: The topological degeneracy Given a many-body ground state

$$|\Omega\rangle \equiv |\mathbb{1}\rangle \quad (3.52a)$$

of the interaction $\hat{\mathcal{H}}_{\text{bs}}$ defined in Eq. (3.9), we can obtain two additional many-body states by acting with the

erator products $\hat{\Gamma}_1^\sigma(t, z) \hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ and $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon) \hat{\Gamma}_1^\sigma(t, z)$ in the same limit. As discussed in Appendix C, the reason for this unusual limiting behavior as a function of the regulator ϵ has to do with the fact that $\hat{\Gamma}_1^\sigma(t, z)$ and $\hat{\Gamma}_{2,y'}^\sigma(t, z_1, \epsilon)$ are nonlocal, and nonunitary, operators that change the topological sectors of the states on which they act, and is inextricably related to the non-Abelian nature of the topological phase. To keep careful track of this limiting behavior, we will always defer the evaluation of the limit $\epsilon \rightarrow 0$ to the end of all calculations.

σ -string operators along the y - and z -directions, respectively,

$$|\hat{\Gamma}_1^\sigma\rangle := \hat{\Gamma}_1^\sigma(z) |\Omega\rangle \quad (3.52b)$$

and

$$|\hat{\Gamma}_2^\sigma\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) |\Omega\rangle, \quad (3.52c)$$

for any z , y' , and z_1 . It is important to point out that not all choices of $|\Omega\rangle$ are equal. As argued in Appendix C, depending on the topological sector in which the state $|\Omega\rangle$ resides, one or both of the states (3.52b) and (3.52c) could have norm zero or infinity. We will first prove that the many-body states $|\hat{\Gamma}_1^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ share the same eigenvalue of $\hat{\mathcal{H}}_{\text{bs}}$ as $|\Omega\rangle$. Second, we will prove that the many-body states (3.52) are linearly independent. In doing so, we will have established that the ground state degeneracy on the torus of the interaction $\hat{\mathcal{H}}_{\text{bs}}$ is threefold.

First, we recall that $\hat{\Gamma}_1^\sigma(z)$ commutes with the interaction $\hat{\mathcal{H}}_{\text{bs}}$ defined in Eq. (3.9). Hence, the many-body state $|\hat{\Gamma}_1^\sigma\rangle$ defined in Eq. (3.52b) is a ground state of the interaction $\hat{\mathcal{H}}_{\text{bs}}$. Making sure to treat the limit $\epsilon \rightarrow 0$ with care, we show in Appendix C that the many-body state $|\hat{\Gamma}_2^\sigma\rangle$ defined in Eq. (3.52b) is also a ground state of the interaction $\hat{\mathcal{H}}_{\text{bs}}$. Now, we are going to show that the three many-body states (3.52) are linearly independent.

The operators $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$ commute with the interaction $\hat{\mathcal{H}}_{\text{bs}}$ and with each other [recall Eq. (3.24)]. They are thus simultaneously diagonalizable. Consequently, we can choose $|\Omega\rangle$ to be a simultaneous eigenstate of the pair of operators $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$. Both $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$ are unitary, i.e., there should exist the pair of unimodular complex numbers $\omega_1^\psi \neq 0$ and $\omega_2^\psi \neq 0$ such that

$$\hat{\Gamma}_1^\psi |\Omega\rangle = \omega_1^\psi |\Omega\rangle, \quad (3.53a)$$

and

$$\hat{\Gamma}_2^\psi |\Omega\rangle = \omega_2^\psi |\Omega\rangle, \quad (3.53b)$$

respectively.

Because of the anticommutator (3.21), we find the eigenvalue

$$\hat{\Gamma}_2^\psi |\hat{\Gamma}_1^\sigma\rangle = -\omega_2^\psi |\hat{\Gamma}_1^\sigma\rangle. \quad (3.54)$$

Hence, $|\Omega\rangle$ and $|\hat{\Gamma}_1^\sigma\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}_2^\psi$ with distinct eigenvalues. As such, $|\Omega\rangle$ and $|\hat{\Gamma}_1^\sigma\rangle$ are orthogonal. Similarly, because of the anticommutator (3.22), we find the eigenvalue

$$\hat{\Gamma}_1^\psi |\hat{\Gamma}_2^\sigma\rangle = -\omega_1^\psi |\hat{\Gamma}_2^\sigma\rangle. \quad (3.55)$$

Hence, $|\Omega\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}_1^\psi$ with distinct eigenvalues. As such,

$|\Omega\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are orthogonal.

To complete the proof that $|\Omega\rangle$, $|\hat{\Gamma}_1^\sigma\rangle$, and $|\hat{\Gamma}_2^\sigma\rangle$ are linearly independent, it suffices to show that $|\hat{\Gamma}_1^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are orthogonal. Because of the commutator (3.25), we find the eigenvalue

$$\hat{\Gamma}_1^\psi |\hat{\Gamma}_1^\sigma\rangle = +\omega_1^\psi |\hat{\Gamma}_1^\sigma\rangle. \quad (3.56)$$

Hence, $|\hat{\Gamma}_1^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}_1^\psi$ with the pair of distinct eigenvalues $+\omega_1^\psi$ and $-\omega_1^\psi$. As such, $|\hat{\Gamma}_1^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are orthogonal.

We note that the commutator (3.26) could equally well have been used to show that $|\hat{\Gamma}_1^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma\rangle$ are simultaneous eigenstates of the unitary operator $\hat{\Gamma}_2^\psi$ with the pair of distinct eigenvalues $+\omega_2^\psi$ and $-\omega_2^\psi$.

As promised, we have shown that the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$ on the torus is threefold degenerate. \square

It is useful to pause at this stage to interpret this lower bound on the ground state degeneracy and how it comes about. Naively, given two pairs of anticommuting non-local operators, all of which commute with the Hamiltonian, [i.e., given Eqs. (3.21) and (3.22)] there are at most four degenerate ground states. In the case of Kitaev's toric code [90], the dimensionality of the ground state manifold saturates this upper bound. However, in the case of the two-dimensional state of matter that we have constructed here, we argue that this is not the case. The reason for this is intimately related to the nonunitarity of the string operators $\hat{\Gamma}_1^\sigma(z)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$.

In particular, we assert that neither of the naively-expected fourth states, namely

$$|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) |\Omega\rangle, \quad (3.57a)$$

and

$$|\hat{\Gamma}_2^\sigma \hat{\Gamma}_1^\sigma\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \hat{\Gamma}_1^\sigma(z) |\Omega\rangle, \quad (3.57b)$$

belongs to the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$. Note that the limit $\epsilon \rightarrow 0$ above is to be taken after forming the products $\hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \hat{\Gamma}_1^\sigma(z)$, as discussed in Footnote 1 and Appendix C. If the operator products $\hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \hat{\Gamma}_1^\sigma(z)$ were to commute with the interaction $\hat{\mathcal{H}}_{\text{bs}}$ in the limit $\epsilon \rightarrow 0$, as they would in an Abelian topological phase, then there would be no obstruction to the states $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle$ and $|\hat{\Gamma}_2^\sigma \hat{\Gamma}_1^\sigma\rangle$ belonging to the ground-state manifold. The proof that such an obstruction exists in the present (non-Abelian) case is undertaken in two complementary ways in the present work. The first, which we call the “algebraic” approach, relies on diagrammatic techniques developed in Appendix D, and is presented below. The second, which we call the “analytic” ap-

proach, is carried out in Appendix C. Both the “algebraic” and “analytic” proofs rely on the fact, discussed in Appendix C, that the operator products $\hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \hat{\Gamma}_1^\sigma(z)$ are not bound to commute with the interaction $\hat{\mathcal{H}}_{\text{bs}}$ in the limit $\epsilon \rightarrow 0$. We now proceed with the “algebraic” version of the proof, and refer the reader to Appendices D and C for more details.

Proof (“algebraic”). We introduce the projection operator

$$\begin{aligned} \hat{\mathcal{P}}_{\text{GSM}} := & \mathcal{N}_1^{-1} |\mathbb{1}\rangle \langle \mathbb{1}| \\ & + \mathcal{N}_{\hat{\Gamma}_1^\sigma}^{-1} |\hat{\Gamma}_1^\sigma\rangle \langle \hat{\Gamma}_1^\sigma| \\ & + \mathcal{N}_{\hat{\Gamma}_2^\sigma}^{-1} |\hat{\Gamma}_2^\sigma\rangle \langle \hat{\Gamma}_2^\sigma| \\ & + \dots \end{aligned} \quad (3.58)$$

onto the ground state manifold. Here, \mathcal{N}_1 is the squared norm of the state $|\mathbb{1}\rangle \equiv |\Omega\rangle$, $\mathcal{N}_{\hat{\Gamma}_1^\sigma}$ is the squared norm of the state $|\hat{\Gamma}_1^\sigma\rangle$, $\mathcal{N}_{\hat{\Gamma}_2^\sigma}$ is the squared norm of the state $|\hat{\Gamma}_2^\sigma\rangle$, and \dots is a sum over any remaining elements from the orthonormal basis of the ground state manifold. By definition, any one of the three states $|\mathbb{1}\rangle$, $|\hat{\Gamma}_1^\sigma\rangle$, and $|\hat{\Gamma}_2^\sigma\rangle$ defined in Eq. (3.52) is invariant under the action of

$$\hat{\mathcal{P}}_{\text{GSM}} = \hat{\mathcal{P}}_{\text{GSM}}^2. \quad (3.59)$$

Hence, we may write

$$|\hat{\Gamma}_1^\sigma\rangle = \hat{\mathcal{P}}_{\text{GSM}} |\hat{\Gamma}_1^\sigma\rangle = \hat{\mathcal{P}}_{\text{GSM}} \hat{\Gamma}_1^\sigma(z) \hat{\mathcal{P}}_{\text{GSM}} |\Omega\rangle, \quad (3.60a)$$

$$|\hat{\Gamma}_2^\sigma\rangle = \hat{\mathcal{P}}_{\text{GSM}} |\hat{\Gamma}_2^\sigma\rangle = \hat{\mathcal{P}}_{\text{GSM}} \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \hat{\mathcal{P}}_{\text{GSM}} |\Omega\rangle. \quad (3.60b)$$

On the other hand,

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\mathcal{O}} \hat{\mathcal{P}}_{\text{GSM}} = 0 \quad (3.61)$$

must hold for any operator $\hat{\mathcal{O}}$ such that $\hat{\mathcal{O}}$ returns an excited state when applied to any state from the ground-state manifold.

We are first going to show that the operators $\hat{\Gamma}_1^\sigma(z)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ do not commute in the limit $\epsilon \rightarrow 0$. After that, we will elaborate on why the state $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle$ does not

belong to the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$.

We begin by considering the exchange algebra of the string operators $\hat{\Gamma}_1^\sigma(z)$ and $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ defined in Eqs. (3.39) and (3.48), respectively. Specifically, we consider the product

$$\begin{aligned} \hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \propto & \left(\prod_{y=0}^{L_y} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z) \right) \\ & \times \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_1 + \epsilon) \hat{\mathcal{P}}_1, \end{aligned} \quad (3.62)$$

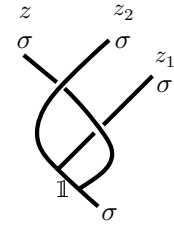
where $\epsilon > 0$ is infinitesimal and we have also omitted the operator $\hat{\mathcal{U}}_{\alpha_{R,y'}=\pi}$ appearing in the definition (3.48), as this operator acts only on the bosonic sector of the theory and thus commutes with all operators in the Ising sector. Using the fact that twist operators in different wires (and in different chiral sectors of the same wire) commute, we deduce that

$$\begin{aligned} \hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon) \propto & \left(\prod_{y \neq y'} \hat{\sigma}_{L,y}(z) \hat{\sigma}_{R,y}(z) \right) \hat{\sigma}_{L,y'}(z) \\ & \times \hat{\sigma}_{R,y'}(z) \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_1 + \epsilon) \hat{\mathcal{P}}_1. \end{aligned} \quad (3.63)$$

Since all operators in the first line of the right-hand side above commute with all operators in the second line, computing the exchange algebra of the operators $\hat{\Gamma}_1^\sigma$ and $\hat{\Gamma}_2^\sigma$ boils down to considering the following product of operators,

$$\lim_{z_2 \rightarrow z_1 + \epsilon} \hat{\sigma}_{R,y'}(z) \hat{\mathcal{P}}_1 \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2) \hat{\mathcal{P}}_1. \quad (3.64)$$

Using the prescriptions of Appendix D, we find that the process of commuting the leftmost operator, $\hat{\sigma}_{R,y'}(z)$, past the remaining two operators is represented by the diagram



$$. \quad (3.65)$$

Untwisting the legs of this fusion diagram, we find

$$\begin{aligned}
\text{Diagram 1} &= \sum_{a,b,c=\mathbb{1},\psi} [F_{\sigma}^{\sigma\sigma\sigma}]_{\mathbb{1}a} (R_a^{\sigma\sigma})^{-1} [F_{\sigma}^{\sigma\sigma\sigma}]_{ab}^{-1} R_b^{\sigma\sigma} [F_{\sigma}^{\sigma\sigma\sigma}]_{bc} \text{Diagram 2} \\
&= e^{+i\frac{\pi}{4}} \text{Diagram 3},
\end{aligned} \tag{3.66}$$

where the F - and R -symbols are given in Appendix D. The diagrammatic relation expressed in Eq. (3.66) can be rewritten as the algebraic statement

$$\begin{aligned}
&\hat{\sigma}_{R,y'}(z) \hat{\mathcal{P}}_{\mathbb{1}} \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2) \hat{\mathcal{P}}_{\mathbb{1}} \\
&= e^{+i\frac{\pi}{4}} \hat{\mathcal{P}}_{\psi} \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2) \hat{\mathcal{P}}_{\psi} \hat{\sigma}_{R,y'}(z),
\end{aligned} \tag{3.67}$$

where $\hat{\mathcal{P}}_{\psi}$ is a projection operator that projects the product $\hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_2)$ into the fusion channel $\sigma \times \sigma = \psi$. Taking the limit $z_2 \rightarrow z_1 + \epsilon$ and restoring the operators $\hat{\sigma}_{M,y}(z)$ present in Eq. (3.63) (as well as the operator $\hat{\mathcal{U}}_{\alpha_{R,y'}=\pi}$ that was omitted there), we arrive at the relation

$$\hat{\Gamma}_1^{\sigma}(z) \hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) = e^{+i\frac{\pi}{4}} \hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) \hat{\Gamma}_1^{\sigma}(z), \tag{3.68a}$$

in the limit $\epsilon \rightarrow 0$, where we have defined the operator

$$\hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) := \hat{\mathcal{U}}_{\alpha_{R,y'}=\pi} \hat{\mathcal{P}}_{\psi} \hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_1 + \epsilon) \hat{\mathcal{P}}_{\psi}, \tag{3.68b}$$

which is identical to the operator $\hat{\Gamma}_2^{\sigma}$ defined in Eq. (3.48), except that the product $\hat{\sigma}_{R,y'}(z_1) \hat{\sigma}_{R,y'}(z_1 + \epsilon)$ is evaluated in the fusion channel ψ rather than the fusion channel $\mathbb{1}$. This difference is fundamental. Since the two twist operators entering the operator $\hat{\Gamma}_2^{\sigma}$ fuse to ψ , this operator can be interpreted as adding an extra Majorana fermion to the state on which it acts. Acting with $\hat{\Gamma}_2^{\sigma}$ on any of the states $|\mathbb{1}\rangle, |\hat{\Gamma}_1^{\sigma}\rangle, |\hat{\Gamma}_2^{\sigma}\rangle, \dots$ in the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$ can then be viewed as creating an *excited state* of the interaction $\hat{\mathcal{H}}_{\text{bs}}$ with one extra fermion. In other words, we have

$$\hat{\mathcal{P}}_{\text{GSM}} \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) \hat{\mathcal{P}}_{\text{GSM}} = 0. \tag{3.69}$$

This relation is crucial in what follows.

We are now prepared to exclude the state $|\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle$ from the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$. Applying Eq. (3.68a) to the definition (3.57a) of the state

	$\hat{\Gamma}_1^{\psi}$	$\hat{\Gamma}_2^{\psi}$	$\hat{\Gamma}_1^{\sigma}$	$\hat{\Gamma}_2^{\sigma}$
$\hat{\Gamma}_1^{\sigma}$	+	-	+	\times
$\hat{\Gamma}_2^{\sigma}$	-	+	\times	+

TABLE I. Summary of the algebra of the string operators $\hat{\Gamma}_{1,2}^{\psi}$ and $\hat{\Gamma}_{1,2}^{\sigma}$. Entries corresponding to a pair of operators that commute are labeled with a +. Entries corresponding to a pair of operators that anticommute are labeled with a -. Entries corresponding to a pair of operators that neither commute nor anticommute are labeled with a \times .

$|\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle$, we obtain

$$|\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle = e^{+i\frac{\pi}{4}} \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) |\hat{\Gamma}_1^{\sigma}\rangle. \tag{3.70}$$

If the state $|\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle$ is in the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$, then it cannot be a null vector of $\hat{\mathcal{P}}_{\text{GSM}}$. However, using Eqs. (3.60) and (3.69), we find that

$$\begin{aligned}
\hat{\mathcal{P}}_{\text{GSM}} |\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle &= e^{+i\frac{\pi}{4}} \hat{\mathcal{P}}_{\text{GSM}} \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y'}^{\sigma}(z_1, \epsilon) \hat{\mathcal{P}}_{\text{GSM}} |\hat{\Gamma}_1^{\sigma}\rangle \\
&= 0.
\end{aligned} \tag{3.71}$$

Thus, the state $|\hat{\Gamma}_1^{\sigma} \hat{\Gamma}_2^{\sigma}\rangle$ does not lie in the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}$. Similarly, the state $|\hat{\Gamma}_2^{\sigma} \hat{\Gamma}_1^{\sigma}\rangle$ defined in Eq. (3.57b) is excluded from the ground-state manifold. We note in passing that a related line of reasoning was used in Ref. [71] to exclude certain states from the ground-state manifold of the gauged $p + ip$ superconductor (see also Ref. [91]). \square

In summary, we have shown that the (2+1)-dimensional $su(2)_2$ coupled-wire construction has a threefold topological degeneracy on the two-torus. The proof that this topological degeneracy is threefold and not fourfold relied on the observation that the σ -string operators obey the non-Abelian exchange algebra (3.68a). This algebra, whereby exchanging the two operators does not simply produce a phase factor, but instead enacts a nontrivial transformation on the operators themselves, is the essence of what it means to be a non-

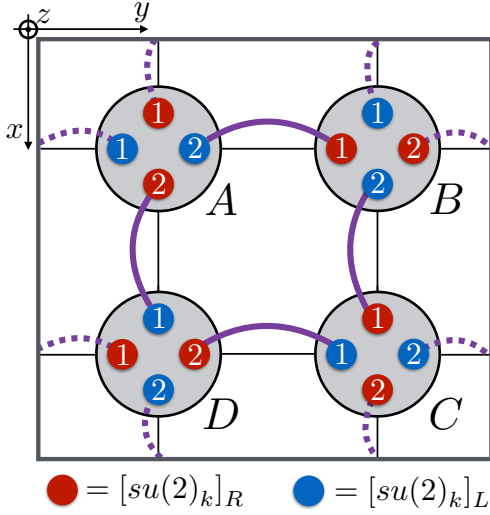


FIG. 6. (Color online) Schematic of the couplings between chiral $SU(2)$ sectors in a four-wire unit cell.

Abelian topological phase. We will see that a similar, albeit richer, algebra arises in the (3+1)-dimensional case discussed in the next section. For future comparison with the (3+1)-dimensional case, we summarize the exchange algebra of the ψ - and σ -string operators in Table I. In both the (2+1)- and (3+1)-dimensional cases, the non-Abelian algebra that encodes the topological degeneracy is induced by the algebra of the primary operators of the corresponding CFT.

IV. NON-ABELIAN TOPOLOGICAL ORDER IN THREE DIMENSIONS

In this section, we generalize the class of (2+1)-dimensional models defined in Sec. III to (3+1) dimensions. The resulting gapped states of matter exhibit non-Abelian topological order enriched by a symmetry analogous to time-reversal. We illustrate the existence and non-Abelian nature of this topological order by considering the $su(2)_2$ case in detail.

A. Definition of the class of models

Consider a square lattice Λ of wires, each described by the Lagrangian density (2.1). We want to break

the degrees of freedom in any one of the identical wires up into four groups, two of which contain only right-moving degrees of freedom and two of which contain only left-moving degrees of freedom (see Fig. 2). Consequently, let each wire (2.1) contain $N_c = 2k$ colors of fermions, so that the full symmetry group of each wire is $U(4k)_L \times U(4k)_R$. However, we wish to employ the conformal embedding (2.3) to write down the couplings in our theory in terms of currents. Thus, let us consider only couplings that are symmetric under the subgroup $U(2k)_M \times U(2k)_M \subset U(4k)_M$ with $M = L, R$. [Note that the central charges associated with the groups $U(2k)_M \times U(2k)_M$ and $U(4k)_M$ are identical [84]. Thus, we can use couplings with either symmetry to fully gap the theory.] Then, we can use the identity

$$u(2k)_1 = u(1) \oplus su(2)_k \oplus su(k)_2 \quad (4.1)$$

to define the M-moving chiral currents \hat{j}_M , \hat{j}_M^a , and \hat{J}_M^a , which are given by Eqs. (2.4) with the substitution $N_c \rightarrow k$. Because we are considering couplings that are symmetric under rotations in $[U(2k) \times U(2k)]_L \times [U(2k) \times U(2k)]_R$, there are actually two copies of each of the chiral currents \hat{j}_M , \hat{j}_M^a , and \hat{J}_M^a with $M = L, R$ in each wire. Therefore, we adopt an additional label $\gamma = 1, 2$ to distinguish the chiral currents $\hat{j}_{\gamma,M}$, $\hat{j}_{\gamma,M}^a$, and $\hat{J}_{\gamma,M}^a$ from one another. The label γ is somewhat redundant in that it will always transform trivially under all the symmetries that we shall impose – we only use it to keep track of the two independent copies of each set of currents.

Next, as in Sec. III A, we gap out the $u(1)$ and $su(k)_2$ degrees of freedom by turning on intra-wire interactions of the form (2.8) and (2.9), respectively, for each $\gamma = 1, 2$. The remaining $su(2)_k$ degrees of freedom are coupled in the following way. First, we define a square lattice $\tilde{\Lambda}$, whose unit cell is enlarged with respect to that of the square lattice Λ . Let the unit cell of $\tilde{\Lambda}$ contain four quantum wires, which we label by an index $J = A, B, C, D$. This enlarged unit cell is depicted in Fig. 6. Each $su(2)_k$ current operator $\hat{J}_{\gamma,M,J,\tilde{r}}^a$ then carries the labels $\gamma = 1, 2$ and $M = L, R$, as well as a label $\tilde{r} \in \tilde{\Lambda}$ to specify the unit cell and a label $J = A, B, C, D$ to specify a wire within a unit cell. We then write down the many-body “backscattering” current-current interactions encoded by the Lagrangian density

$$\begin{aligned}
 \hat{\mathcal{L}}_{\text{bs}}[su(2)_k] \equiv -\hat{\mathcal{H}}_{\text{bs}}[su(2)_k] := & -\lambda_{su(2)_k} \sum_{\tilde{r} \in \tilde{\Lambda}} \sum_{a=1}^3 \left(\hat{\mathcal{L}}_{2,R,A,\tilde{r}|1,L,D} + \hat{\mathcal{L}}'_{2,L,D,\tilde{r}|1,R,A} + \hat{\mathcal{L}}_{2,L,B,\tilde{r}|1,R,C} + \hat{\mathcal{L}}'_{2,R,C,\tilde{r}|1,L,B} \right. \\
 & \left. + \hat{\mathcal{L}}_{2,L,A,\tilde{r}|1,R,B} + \hat{\mathcal{L}}''_{2,R,B,\tilde{r}|1,L,A} + \hat{\mathcal{L}}_{2,R,D,\tilde{r}|1,L,C} + \hat{\mathcal{L}}''_{2,L,C,\tilde{r}|1,R,D} \right), \quad (4.2a)
 \end{aligned}$$

where we have assigned to each of the eight nearest-neighbor bonds shown in Fig. 6 the bond operators

$$\hat{\mathcal{L}}_{\gamma,M,J,\tilde{\mathbf{r}}|\gamma',M',J'} := \sum_{a=1}^3 \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^a \hat{J}_{\gamma',M',J',\tilde{\mathbf{r}}}^a, \quad (4.2b)$$

$$\hat{\mathcal{L}}'_{\gamma,M,J,\tilde{\mathbf{r}}|\gamma',M',J'} := \sum_{a=1}^3 \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^a \hat{J}_{\gamma',M',J',\tilde{\mathbf{r}}+2\hat{\mathbf{x}}}^a, \quad (4.2c)$$

$$\hat{\mathcal{L}}''_{\gamma,M,J,\tilde{\mathbf{r}}|\gamma',M',J'} := \sum_{a=1}^3 \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^a \hat{J}_{\gamma',M',J',\tilde{\mathbf{r}}+2\hat{\mathbf{y}}}^a, \quad (4.2d)$$

and where the lattice vectors $2\hat{\mathbf{x}}$ and $2\hat{\mathbf{y}}$ connect neighboring unit cells along the x - and y -directions, respectively. Like the current-current interactions used to couple neighboring $su(2)_k$ modes in the (2+1)-dimensional case [see Eq. (3.2)], the interactions (4.2a) are marginally relevant, flowing to strong coupling for $\lambda_{su(2)_k} > 0$. Since all right- and left-handed currents are paired by current-current interactions when periodic boundary conditions are imposed in all directions, we conclude that the local inter-wire many-body interactions yield a fully gapped bulk.

The current-current interactions (4.2a) possess a symmetry that plays the role of time-reversal symmetry in the coupled-wire model. To see this, note that Figure 6 is invariant under interchanging the colors red and blue, and then translating by either of the *half*-lattice vectors $\hat{\mathbf{x}}$ or $\hat{\mathbf{y}}$. Formally, we define the symmetry operations

$$\mathcal{T}_{\text{eff},\hat{\mathbf{x}}} := \mathcal{T} \times \mathbb{T}_{\hat{\mathbf{x}}} \quad (4.3a)$$

$$\mathcal{T}_{\text{eff},\hat{\mathbf{y}}} := \mathcal{T} \times \mathbb{T}_{\hat{\mathbf{y}}}, \quad (4.3b)$$

where the time-reversal operation \mathcal{T} acts in the usual way on the spinful fermions (i.e., $\mathcal{T}^2 = -1$), and acts on the $su(2)_k$ currents [see Eqs. (2.4)] as

$$\mathcal{T} \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^a \mathcal{T}^{-1} = -\hat{J}_{\gamma,\bar{M},J,\tilde{\mathbf{r}}}^a, \quad (4.3c)$$

for any $a = 1, 2, 3$, $\gamma = 1, 2$, $M = R, L$, $J = A, B, C, D$, and $\tilde{\mathbf{r}} \in \tilde{\Lambda}$, with $\bar{L} := R$ and $\bar{R} := L$, and where the half-lattice translation operators $\mathbb{T}_{\hat{\mathbf{x}}}$ and $\mathbb{T}_{\hat{\mathbf{y}}}$ act as

$$\mathbb{T}_{\hat{\mathbf{x}}} \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}} \mathbb{T}_{\hat{\mathbf{x}}}^{-1} = \begin{cases} \hat{J}_{\gamma,M,B,\tilde{\mathbf{r}}} & J = A \\ \hat{J}_{\gamma,M,A,\tilde{\mathbf{r}}+2\hat{\mathbf{x}}} & J = B \\ \hat{J}_{\gamma,M,D,\tilde{\mathbf{r}}+2\hat{\mathbf{x}}} & J = C \\ \hat{J}_{\gamma,M,C,\tilde{\mathbf{r}}} & J = D \end{cases}, \quad (4.3d)$$

and

$$\mathbb{T}_{\hat{\mathbf{y}}} \hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}} \mathbb{T}_{\hat{\mathbf{y}}}^{-1} = \begin{cases} \hat{J}_{\gamma,M,D,\tilde{\mathbf{r}}} & J = A \\ \hat{J}_{\gamma,M,C,\tilde{\mathbf{r}}} & J = B \\ \hat{J}_{\gamma,M,B,\tilde{\mathbf{r}}+2\hat{\mathbf{y}}} & J = C \\ \hat{J}_{\gamma,M,A,\tilde{\mathbf{r}}+2\hat{\mathbf{y}}} & J = D \end{cases}, \quad (4.3e)$$

respectively. The full Lagrangian in the presence of the current-current interactions (4.2a) is invariant under the symmetry operations $\mathcal{T}_{\text{eff},\hat{\mathbf{x}}}$ and $\mathcal{T}_{\text{eff},\hat{\mathbf{y}}}$ when periodic boundary conditions are imposed in the x - and y -directions. If we use mixed boundary conditions, such as ones that are open along the x -direction and periodic along the y -direction (or vice versa), then only the symmetry $\mathcal{T}_{\text{eff},\hat{\mathbf{y}}}$ (or $\mathcal{T}_{\text{eff},\hat{\mathbf{x}}}$) remains. In Sec. V, we will see that the remaining symmetry $\mathcal{T}_{\text{eff},\hat{\mathbf{y}}}$ protects gapless surface states on the boundaries at $x = 0$ and $x = L_x$. Analogous non-onsite implementations of time reversal symmetry have arisen in studies of antiferromagnetic TIs [74, 75] and in coupled-wire models of topological-insulator and topological-superconductor surfaces [72, 73]; we expect that there exists an analogue of the gapped (3+1)-dimensional phase constructed in this work in which time-reversal acts in the usual onsite manner.

B. Parafermion representation of the interwire interactions

The starting point for our analysis of the bulk excitations is to notice that the interactions (4.2a) that open a gap in the bulk can be rewritten using the identities (3.4). In particular, we decompose the $su(2)_k$ currents as [84]

$$\hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^+ = \sqrt{k} \hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}} : e^{+i\sqrt{1/k}\hat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}} :, \quad (4.4a)$$

$$\hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^- = \sqrt{k} \hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}^\dagger : e^{-i\sqrt{1/k}\hat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}} :, \quad (4.4b)$$

$$\hat{J}_{\gamma,M,J,\tilde{\mathbf{r}}}^3 = i\frac{\sqrt{k}}{2} \partial_M \hat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}, \quad (4.4c)$$

into the pair of chiral parafermionic quantum fields $\hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}^\dagger$ and $\hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}$ together with the chiral bosonic quantum field $\hat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}$ for any $\gamma = 1, 2$, $M = L, R$, $J = A, B, C, D$. On the parafermions, we impose the equal-time algebra

$$\begin{aligned} \hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t, z) \hat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}(t, z') &= \hat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}(t, z') \hat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t, z) \\ &\times e^{-i\frac{2\pi}{k} \delta_{\mathbf{r},\mathbf{r}'} \left[(-1)^M \delta_{M,M'} \delta_{\gamma,\gamma'} \text{sgn}(z-z') + (-1)^{\gamma+\gamma'} \text{sgn}((\gamma,M) - (\gamma',M')) \right]} \\ &\times e^{+i\frac{2\pi}{k} [\text{sgn}(y-y') + \delta_{y,y'} \text{sgn}(x-x')]}, \end{aligned} \quad (4.5a)$$

$$\begin{aligned}
\widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \widehat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}^\dagger(t,z') &= \widehat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}^\dagger(t,z') \widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \\
&\times e^{-i \frac{2\pi}{k} \delta_{\mathbf{r},\mathbf{r}'} \left[(-1)^M \delta_{M,M'} \delta_{\gamma,\gamma'} \text{sgn}(z-z') + (-1)^{\gamma+\gamma'} \text{sgn}((\gamma,M) - (\gamma',M')) \right]} \\
&\times e^{+i \frac{2\pi}{k} [\text{sgn}(y-y') + \delta_{y,y'} \text{sgn}(x-x')]},
\end{aligned} \tag{4.5b}$$

$$\begin{aligned}
\widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \widehat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}^\dagger(t,z') &= \widehat{\Psi}_{\gamma',M',J',\tilde{\mathbf{r}}'}^\dagger(t,z') \widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \\
&\times e^{+i \frac{2\pi}{k} \delta_{\mathbf{r},\mathbf{r}'} \left[(-1)^M \delta_{M,M'} \delta_{\gamma,\gamma'} \text{sgn}(z-z') + (-1)^{\gamma+\gamma'} \text{sgn}((\gamma,M) - (\gamma',M')) \right]} \\
&\times e^{-i \frac{2\pi}{k} [\text{sgn}(y-y') + \delta_{y,y'} \text{sgn}(x-x')]}
\end{aligned} \tag{4.5c}$$

This equal-time algebra is a generalization of Eqs. (3.4e)–(3.4g). As before, we use the convention $\text{sgn}(0) = 0$. The quantity $\text{sgn}((\gamma, M) - (\gamma', M'))$ is to be evaluated according to the (arbitrary) ordering $(1, R) < (1, L) < (2, L) < (2, R)$. Note that, on the one hand, the parafermion operator $\widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}$ is labeled by the coordinate $\tilde{\mathbf{r}} \in \tilde{\Lambda}$, which is the square lattice with a four-wire unit cell depicted in Fig. 6. On the other hand, the coordinates $\mathbf{r} := (x, y) \in \Lambda$, which is the square lattice with a single-wire unit cell, enter the phase factors in Eqs. (4.5a)–(4.5c). This notation is consistent because the two labels $\tilde{\mathbf{r}}$ and $J = A, B, C, D$ uniquely specify an $\mathbf{r} \in \Lambda$. Finally, we impose the equal-time algebra

$$\begin{aligned}
\left[\widehat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z), \widehat{\phi}_{\gamma',M',J',\tilde{\mathbf{r}}'}(t,z') \right] &= -i 2\pi \left[(-1)^M \delta_{M,M'} \delta_{\gamma,\gamma'} \delta_{\mathbf{r},\mathbf{r}'} \text{sgn}(z-z') + \delta_{\mathbf{r},\mathbf{r}'} (-1)^{\gamma+\gamma'} \text{sgn}((\gamma, M) - (\gamma', M')) \right. \\
&\quad \left. - \text{sgn}(y-y') - \delta_{y,y'} \text{sgn}(x-x') \right],
\end{aligned} \tag{4.5d}$$

on the chiral bosons. This algebra is a generalization of Eq. (3.4h).

The interaction that we shall consider was defined in Eq. (4.2a). With the definitions (4.4), the interaction connecting wires A and B within the unit cell labeled by $\tilde{\mathbf{r}} \in \tilde{\Lambda}$ (see Fig. 6) is represented by

$$\widehat{\mathcal{L}}_{2,L,A,\tilde{\mathbf{r}}|1,R,B} = \frac{k}{2} \left[\left(e^{+i\sqrt{1/k}(\widehat{\phi}_{2,L,A,\mathbf{r}} - \widehat{\phi}_{1,R,B,\tilde{\mathbf{r}}})} \widehat{\Psi}_{1,R,B,\tilde{\mathbf{r}}}^\dagger \widehat{\Psi}_{2,L,A,\tilde{\mathbf{r}}} + \text{H.c.} \right) - \frac{1}{2} \left(\partial_L \widehat{\phi}_{2,L,A,\tilde{\mathbf{r}}} \right) \left(\partial_R \widehat{\phi}_{1,R,B,\tilde{\mathbf{r}}} \right) \right]. \tag{4.6}$$

Similar expressions hold for the remaining seven couplings depicted in Fig. 6.

C. Case study: $su(2)_2$

To illustrate how to investigate the nature of the bulk topological phase resulting from the interactions (4.6), let us consider the case $k = 2$. Our discussion will build on

the contents of Sec. III C, where a two-dimensional version of this model was considered in detail. With $k = 2$, the chiral parafermion operators become the chiral Majorana fermion operators

$$\widehat{\psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \equiv \widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) = \widehat{\Psi}_{\gamma,M,J,\tilde{\mathbf{r}}}^\dagger(t,z) \tag{4.7}$$

where z_M is defined in Eq. (2.2) and $\gamma = 1, 2$, $M = L, R$, $J = A, B, C, D$. They satisfy the equal-time algebra (4.5a) with $k = 2$. The local interaction (4.6) is denoted

$$\widehat{\mathcal{L}}_{2,L,A,\tilde{\mathbf{r}}|1,R,B} = \left[\left(e^{+i\sqrt{1/2}(\widehat{\phi}_{2,L,A,\mathbf{r}} - \widehat{\phi}_{1,R,B,\tilde{\mathbf{r}}})} \widehat{\psi}_{1,R,B,\tilde{\mathbf{r}}} \widehat{\psi}_{2,L,A,\tilde{\mathbf{r}}} + \text{H.c.} \right) - \frac{1}{2} \left(\partial_L \widehat{\phi}_{2,L,A,\tilde{\mathbf{r}}} \right) \left(\partial_R \widehat{\phi}_{1,R,B,\tilde{\mathbf{r}}} \right) \right]. \tag{4.8}$$

We will focus on the task of building a set of nonlocal operators that commute with the eight bond interactions of the form (4.8) and whose eigenvalues can be used to label the ground states of the gapped bulk when periodic boundary conditions are imposed in all directions.

Before proceeding to characterize the bulk topological order in the three-dimensional case, we note that the eight bond interactions of the form (4.8) possess the same quasilocal \mathbb{Z}_2 symmetry as the one explored in the

two-dimensional case discussed in Sec. III C. Namely, the eight bond interactions of the form (4.8) are invariant under the wire- and chirality-resolved transformation

$$\widehat{\psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \mapsto e^{i\alpha_{\gamma,M,J,\tilde{\mathbf{r}}}} \widehat{\psi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z), \tag{4.9a}$$

$$\widehat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) \mapsto \widehat{\phi}_{\gamma,M,J,\tilde{\mathbf{r}}}(t,z) + \sqrt{2} \alpha_{\gamma,M,J,\tilde{\mathbf{r}}}, \tag{4.9b}$$

where the assignments

$$\alpha_{\gamma,M,J,\tilde{\mathbf{r}}} \in \{0, \pi\} \quad (4.9c)$$

for all $\gamma = 1, 2$, $M = L, R$, and all wires $(J, \tilde{\mathbf{r}})$ with $J = A, B, C, D$ and $\tilde{\mathbf{r}} \in \tilde{\Lambda}$ define the map α from $\{\gamma = 1, 2\} \times \{M = L, R\} \times \{J = A, B, C, D\} \times \{\tilde{\mathbf{r}} \in \tilde{\Lambda}\}$ to $\{0, \pi\}$. The generator of this quasilocal \mathbb{Z}_2 gauge transformation can be defined following the arguments of Eqs. (3.14)–(3.13) in Sec. III C. For any choice of the function α , we can define the generator

$$\hat{\Gamma}_\alpha(t) \equiv \prod_{\gamma,M,J,\tilde{\mathbf{r}}} \hat{\Gamma}_{\alpha_{\gamma,M,J,\tilde{\mathbf{r}}}}(t) := \hat{\mathcal{U}}_\alpha(t) \hat{\mathcal{Z}}_\alpha(t), \quad (4.10)$$

where $\hat{\mathcal{U}}_\alpha(t)$, which implements the transformation (4.9b), is defined in direct analogy with Eq. (3.14), and $\hat{\mathcal{Z}}_\alpha(t)$, which implements the transformation (4.9a), is also defined in direct analogy to the (2+1)-dimensional case.

1. Majorana-string and Majorana-membrane operators

String operators for ψ and σ fields are constructed by analogy with the two-dimensional case discussed in Sec. III C. We first show how to construct string operators for the Majorana fields ψ . To construct ψ string operators that act along paths in the x - y plane, we define the bilinear operators

$$\hat{\mathcal{O}}_{\gamma\gamma',J,\tilde{\mathbf{r}}}^\psi(t,z) := \hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(t,z) \hat{\psi}_{\gamma',L,J,\tilde{\mathbf{r}}}(t,z). \quad (4.11)$$

We then define the string operators

$$\hat{\Gamma}_{\hat{x}}^\psi(t,z) := \prod_{(\gamma,\gamma',J,\tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{x}}} \hat{\mathcal{O}}_{\gamma\gamma',J,\tilde{\mathbf{r}}}^\psi(t,z), \quad (4.12a)$$

$$\hat{\Gamma}_{\hat{y}}^\psi(t,z) := \prod_{(\gamma,\gamma',J,\tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{y}}} \hat{\mathcal{O}}_{\gamma\gamma',J,\tilde{\mathbf{r}}}^\psi(t,z). \quad (4.12b)$$

Even though these act along the noncontractible paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, respectively, shown in Fig. 7, we have suppressed their explicit dependence on the choice made for $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$. Note that although the paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ are not orthogonal, any two such paths share exactly two chiral modes, one right-moving and one left-moving, belonging to neighboring wires [see Fig. 7(b)]. One can show that this is the minimal number of times that any two paths in the square lattice Λ can intersect. A calculation analogous to Eqs. (3.29) in Sec. III C shows that the string operators $\hat{\Gamma}_{\hat{x}}^\psi(t,z)$ and $\hat{\Gamma}_{\hat{y}}^\psi(t,z)$ commute with the interaction (4.8) and all other current-current interactions in $\mathcal{L}_{\text{bs}}[su(2)_{k=2}]$ defined in Eq. (4.2a), as long as periodic boundary conditions are imposed along both the \hat{x} - and \hat{y} -directions. If we instead act with any product of the bilinears (4.11) along a path that is not closed, we then create a pair of excitations at the endpoints of

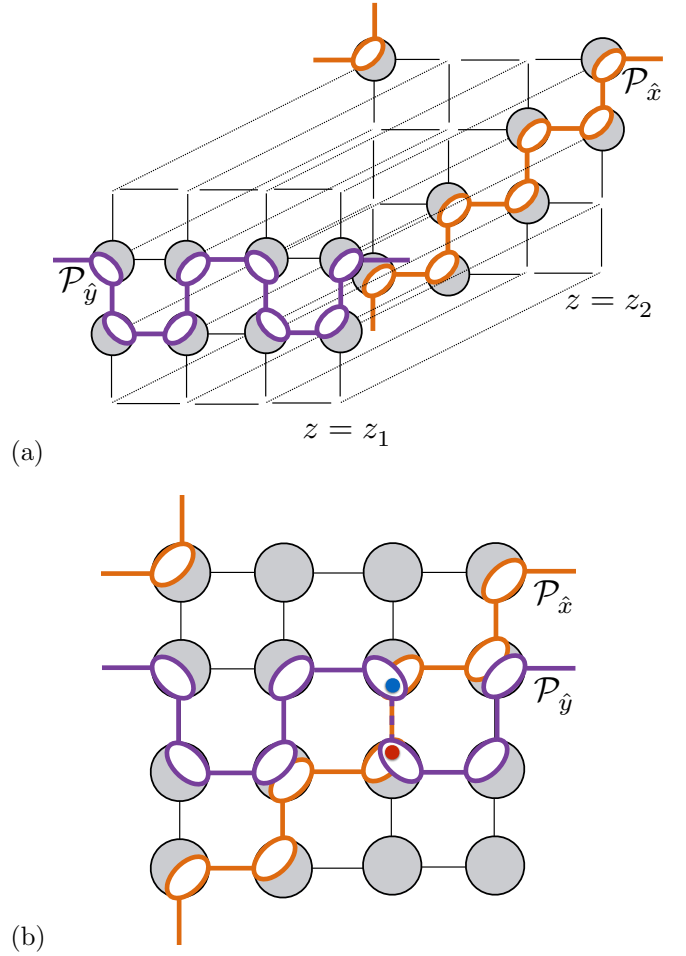


FIG. 7. (Color online) Depictions of the paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$. The grey circles represent wires, as in Fig. 2, and each white oval contains a pair of chiral modes labeled by the indices γ , M , J , and $\tilde{\mathbf{r}}$. In panel (a), the paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ are depicted at different values of the coordinate z , while in panel (b), the projection of the two paths into the x - y plane is shown.

the path. Thus, either one of the operators $\hat{\Gamma}_{\hat{x}}^\psi(z)$ and $\hat{\Gamma}_{\hat{y}}^\psi(z)$ can be viewed as creating a pair of ψ particles, before transporting one of them around the three-torus in a noncontractible loop and re-annihilating the pair. The two paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, along which these operators act, correspond to linearly independent noncontractible cycles of the three-torus.

To construct ψ string operators that act along the z -direction, parallel to the wires, we define the bilinear operators

$$\hat{\mathcal{O}}_{\gamma\gamma',J,J',\tilde{\mathbf{r}}\tilde{\mathbf{r}}'}^\psi(t,z_1,z_2) := \hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(t,z_1) \hat{\psi}_{\gamma',L,J',\tilde{\mathbf{r}}'}(t,z_2), \quad (4.13)$$

where γ , γ' , J , J' , $\tilde{\mathbf{r}}$, and $\tilde{\mathbf{r}}'$ are defined such that $\hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(t,z_1)$ and $\hat{\psi}_{\gamma',L,J',\tilde{\mathbf{r}}'}(t,z_2)$ live on opposite sides of one of the bonds on the square lattice Λ of wires. A calculation analogous to Eq. (3.33) shows that the oper-

ator

$$\hat{\Gamma}_{\hat{z}}^{\psi}(t) := \hat{\mathcal{O}}_{\gamma\gamma', J, J', \tilde{\mathbf{r}}\tilde{\mathbf{r}}'}^{\psi}(t, 0, L_z), \quad (4.14)$$

defined for an arbitrary choice of $\gamma, \gamma', J, J', \tilde{\mathbf{r}},$ and $\tilde{\mathbf{r}}'$ such that $\hat{\psi}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(t, z_1)$ and $\hat{\psi}_{\gamma', \mathbf{L}, J', \tilde{\mathbf{r}}'}(t, z_2)$ live on opposite sides of one of the bonds on the square lattice Λ of wires, commutes with the interaction (4.8) and all other current-current interactions in $\mathcal{L}_{\text{bs}}[su(2)_{k=2}]$ defined in Eq. (4.2a), as long as periodic boundary conditions are imposed along the \hat{z} -direction. [Recall that any integration over z is to be interpreted in the sense of Eq. (3.35).] Similarly to the construction of ψ -string operators that act along paths in the x - y plane, we note that the bilinear $\hat{\mathcal{O}}_{\gamma\gamma', J, J', \tilde{\mathbf{r}}\tilde{\mathbf{r}}'}^{\psi}(t, z_1, z_2)$ generically fails to commute with $\mathcal{L}_{\text{bs}}[su(2)_{k=2}]$ at z_1 and z_2 . Thus, $\hat{\Gamma}_{\hat{z}}^{\psi}(t)$ can be interpreted as creating a pair of ψ particles, before tunneling one of them around the three-torus in the z -direction and annihilating it with its partner.

We now consider how to construct ψ membrane operators. The simplest case is that of a ψ membrane in the x - y plane, which is built by simply acting with the bilinears (4.11) in all wires

$$\hat{\Sigma}_{\hat{z}}^{\psi}(t, z) := \prod_{J=A}^D \prod_{\tilde{\mathbf{r}} \in \Lambda} \hat{\mathcal{O}}_{11, J, \tilde{\mathbf{r}}}^{\psi}(t, z) \hat{\mathcal{O}}_{22, J, \tilde{\mathbf{r}}}^{\psi}(t, z). \quad (4.15)$$

Here, we have adopted the convention whereby any membrane carries the label of a path that is normal to the membrane. Thus, the above membrane in the x - y plane is labeled by \hat{z} . In order to construct a pair of ψ -membranes that are orthogonal to the x - y plane one defines along the paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ the pair of products of the bilinears defined by Eq. (4.13), i.e.,

$$\hat{\Sigma}_{\hat{y}}^{\psi}(t) := \prod_{(\gamma, \gamma', J, J', \tilde{\mathbf{r}}, \tilde{\mathbf{r}}') \in \mathcal{P}_{\hat{x}}} \hat{\mathcal{O}}_{\gamma\gamma', J, J', \tilde{\mathbf{r}}\tilde{\mathbf{r}}'}^{\psi}(t, 0, L_z), \quad (4.16a)$$

$$\hat{\Sigma}_{\hat{x}}^{\psi}(t) := \prod_{(\gamma, \gamma', J, J', \tilde{\mathbf{r}}, \tilde{\mathbf{r}}') \in \mathcal{P}_{\hat{y}}} \hat{\mathcal{O}}_{\gamma\gamma', J, J', \tilde{\mathbf{r}}\tilde{\mathbf{r}}'}^{\psi}(t, 0, L_z). \quad (4.16b)$$

A depiction of the membrane corresponding to Eq. (4.16b) is shown in Fig. 8. These three classes of ψ -membrane operators share the crucial feature that (i) they commute with the current-current interactions in $\mathcal{L}_{\text{bs}}[su(2)_{k=2}]$ defined in Eq. (4.2a), when they act along closed surfaces (as they are defined above), and (ii) they support defects on their boundaries when their constituent bilinears act on some simply connected open subset of a closed surface (as one would obtain by truncating the products in the above definitions). The explicit dependence on the choice for the noncontractible cycles $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ entering in the definitions of the ψ -membrane operators (4.16a) and (4.16b), respectively, has been suppressed.

Using the definitions (4.12), (4.16), and the equal-time algebra (4.5) with $k = 2$, one can show that the ψ -string

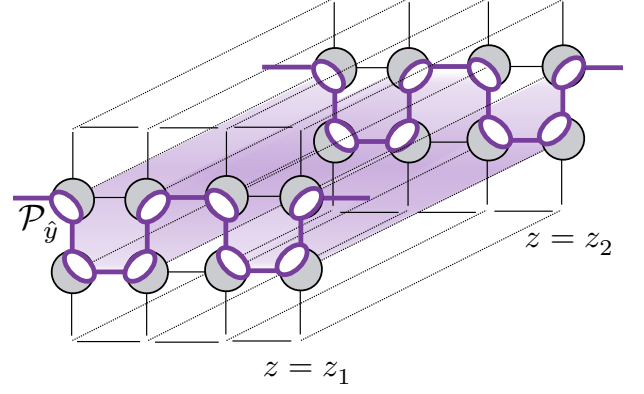


FIG. 8. (Color online) A cartoon representation of a ψ -membrane operator corresponding to substituting 0 by z_1 and L_z by z_2 in Eq. (4.16b).

operators $\hat{\Gamma}_{\hat{a}}^{\psi}$ and ψ -membrane operators $\hat{\Sigma}_{\hat{b}}^{\psi}$ commute for any $\hat{a}, \hat{b} = \hat{x}, \hat{y}, \hat{z}$. Moreover, one verifies that this equal-time algebra is independent of the details of how one defines the paths and surfaces on which the string and membranes act, i.e., deforming the path along which a string operator acts, or the surface on which a membrane operator acts, has no effect on the equal-time algebra as long as these deformations leave the intersection of the path and surface intact. This equal-time algebra can be interpreted as defining the braiding statistics between pointlike and linelike fermionic excitations in the three-dimensional bulk. A similar analysis reveals that two ψ -string operators always commute at equal times. This is consistent with the fact that pointlike fermionic excitations in three spatial dimensions have trivial braiding statistics.

2. Twist-string and twist-membrane operators

String operators corresponding to the Ising twist field σ can be constructed as follows. Similarly to the case of ψ strings, σ strings acting along paths in the x - y plane are built out of the bilinear operators

$$\hat{\mathcal{O}}_{\gamma\gamma', J, \tilde{\mathbf{r}}}^{\sigma}(t, z) := \hat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(t, z) \hat{\sigma}_{\gamma', \mathbf{L}, J, \tilde{\mathbf{r}}}(t, z). \quad (4.17)$$

The σ -string operators themselves are then defined by

$$\hat{\Gamma}_{\hat{x}}^{\sigma}(t, z) := \prod_{(\gamma, \gamma', J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{x}}} \hat{\mathcal{O}}_{\gamma\gamma', J, \tilde{\mathbf{r}}}^{\sigma}(t, z), \quad (4.18a)$$

$$\hat{\Gamma}_{\hat{y}}^{\sigma}(t, z) := \prod_{(\gamma, \gamma', J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{y}}} \hat{\mathcal{O}}_{\gamma\gamma', J, \tilde{\mathbf{r}}}^{\sigma}(t, z). \quad (4.18b)$$

Hereto, we have suppressed the explicit dependence on the choice for the noncontractible cycles $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ en-

tering in the definitions of the σ -string operators (4.18a) and (4.18b), respectively.

One verifies, using Eq. (3.18), that these σ -string operators have the following equal-time algebra with the ψ -membrane operators. Any σ -string operator with a noncontractible cycle in the x - y plane anticommutes with a ψ -membrane operator orthogonal to the noncontractible cycle,

$$\hat{\Gamma}_a^\sigma(t, z) \hat{\Sigma}_a^\psi(t) = -\hat{\Sigma}_a^\psi(t) \hat{\Gamma}_a^\sigma(t, z), \quad (4.19a)$$

for any $\hat{a} = \hat{x}, \hat{y}$. In contrast, any σ -string operator with a noncontractible cycle in the x - y plane commutes with any ψ -membrane operator such that the noncontractible cycle and membrane are not pairwise orthogonal, i.e.

$$\hat{\Gamma}_a^\sigma(t, z) \hat{\Sigma}_b^\psi(t) = \hat{\Sigma}_b^\psi(t) \hat{\Gamma}_a^\sigma(t, z), \quad (4.19b)$$

for $(\hat{a}, \hat{b}) = (\hat{x}, \hat{y})$ or $(\hat{a}, \hat{b}) = (\hat{y}, \hat{x})$. This equal-time algebra holds independently of local deformations of the paths and surfaces on which the string and membrane operators are defined, so long as these deformations leave the intersections of these paths and surfaces unchanged. This equal-time algebra has the interpretation that σ strings can be viewed as operators that twist the boundary conditions of a fermion loop that expands to encompass the entire system along some surface.

To complete the equal-time algebra (4.19), we need a σ string acting along the z -direction. These are constructed by analogy with the two-dimensional case discussed in Sec. III C. We define the operator

$$\hat{\Gamma}_{\hat{z}}^\sigma(z_1, \epsilon, t) := \hat{\mathcal{U}}_{\alpha_{\gamma, M, J, \hat{r}} = \pi} \hat{\sigma}_{\gamma, M, J, \hat{r}}(t, z_1) \hat{\sigma}_{\gamma, M, J, \hat{r}}(t, z_1 + \epsilon), \quad (4.20)$$

where we recall that the operator $\hat{\mathcal{U}}_{\alpha_{\gamma, M, J, \hat{r}}}$, which acts solely on the bosonic sector of the theory, is defined in direct analogy with the operator $\hat{\mathcal{U}}_{\alpha_{M, y}}$ appearing in Eq. (3.14). We have also suppressed the implicit dependence

of the operator $\hat{\Gamma}_{\hat{z}}^\sigma(z_1, \epsilon, t)$ on the choice of indices γ, M, J , and \hat{r} specifying the wire in which it acts. The operator product $\hat{\sigma}_{\gamma, M, J, \hat{r}}(t, z_1) \hat{\sigma}_{\gamma, M, J, \hat{r}}(t, z_1 + \epsilon)$ anticommutes with the Majorana fermion $\hat{\psi}_{\gamma, M, J, \hat{r}}$ in the limit $\epsilon \rightarrow 0$, and commutes with any Majorana fermion $\hat{\psi}_{\gamma', M', J', \hat{r}'}$ with $(\gamma', M', J', \hat{r}') \neq (\gamma, M, J, \hat{r})$. The base point z_1 of the σ string is arbitrary, as is the choice of γ, M, J , and \hat{r} . The definition (4.20) of the operator $\hat{\Gamma}_{\hat{z}}^\sigma(z_1, \epsilon, t)$ is subject to the same caveats as its analogue in one less spatial dimension, which was defined in Eq. (3.48). In particular, the limit $\epsilon \rightarrow 0$ must be taken carefully, as discussed in Footnote 1 and Appendix C. As in the 2D case, we will only take the limit $\epsilon \rightarrow 0$ at the end of calculations.

With these definitions, we can complete the equal-time algebra (4.18) with the following equations. The σ -string operator that acts on a noncontractible cycle oriented along the \hat{z} -direction anticommutes with any ψ -membrane operator acting on a surface that is orthogonal to the \hat{z} -direction,

$$\hat{\Gamma}_{\hat{z}}^\sigma(t, z_1, \epsilon) \hat{\Sigma}_{\hat{z}}^\psi(t, z) = -\hat{\Sigma}_{\hat{z}}^\psi(t, z) \hat{\Gamma}_{\hat{z}}^\sigma(t, z_1, \epsilon), \quad (4.21a)$$

for any infinitesimal $\epsilon > 0$. The σ -string operator acting on a noncontractible cycle oriented along the \hat{z} -direction commutes with any ψ -membrane operator acting on a surface that is orthogonal to the x - or y -directions (for simplicity, we assume that the z -string does not intersect with the x - and y -membranes),

$$\hat{\Gamma}_{\hat{z}}^\sigma(t, z_1, \epsilon) \hat{\Sigma}_a^\psi(t) = \hat{\Sigma}_a^\psi(t) \hat{\Gamma}_{\hat{z}}^\sigma(t, z_1, \epsilon), \quad (4.21b)$$

for any $\hat{a} = \hat{x}, \hat{y}$ and for any infinitesimal $\epsilon > 0$.

Once we have constructed the σ -string operators, we can also investigate the braiding statistics of pointlike particles in the coupled-wire theory. For example, the mutual statistics of σ and ψ excitations can be deduced from exchange relations like

$$\hat{\Gamma}_{\hat{y}}^\psi(t, z) \hat{\Gamma}_{\hat{x}}^\sigma(t, z') = \hat{\Gamma}_{\hat{x}}^\sigma(t, z') \hat{\Gamma}_{\hat{y}}^\psi(t, z) e^{-i \frac{\pi}{2} \text{sgn}(z-z')} e^{+i \frac{\pi}{2} \text{sgn}(z-z')} = \hat{\Gamma}_{\hat{x}}^\sigma(t, z') \hat{\Gamma}_{\hat{y}}^\psi(t, z), \quad (4.22)$$

where we used the counterparts to the equal-time algebra (3.18), which demonstrates that ψ and σ particles braid trivially in the three-dimensional model. Likewise, the self-statistics of σ excitations can be deduced from exchange relations like

$$\begin{aligned} \hat{\Gamma}_{\hat{y}}^\sigma(t, z) \hat{\Gamma}_{\hat{x}}^\sigma(t, z') &= \hat{\Gamma}_{\hat{x}}^\sigma(t, z') \hat{\Gamma}_{\hat{y}}^\sigma(t, z) \times \begin{cases} e^{-i \frac{\pi}{8} \text{sgn}(z-z')} e^{+i \frac{\pi}{8} \text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \mathbb{1}, \\ e^{+i \frac{3\pi}{8} \text{sgn}(z-z')} e^{-i \frac{3\pi}{8} \text{sgn}(z-z')}, & \text{if } \sigma \times \sigma = \psi \end{cases} \\ &= \hat{\Gamma}_{\hat{x}}^\sigma(t, z') \hat{\Gamma}_{\hat{y}}^\sigma(t, z), \end{aligned} \quad (4.23)$$

where we used the counterparts to the equal-time algebra (3.20). The meaning of the two cases distinguished

above, namely the cases $\sigma \times \sigma = \mathbb{1}$ and $\sigma \times \sigma = \psi$, is as follows. When two σ strings act along the noncon-

tractible cycles $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, they necessarily coincide in exactly two chiral channels located astride a bond of the square lattice [see Fig. 7(b)]. Each of these chiral channels is acted upon by two σ operators, one from the $\mathcal{P}_{\hat{x}}$ string and one from the $\mathcal{P}_{\hat{y}}$ string. The outcome of fusing the two σ fields in each of the two channels is correlated. If one pair of σ s fuses to $\mathbb{1}$ or ψ , then the other pair must fuse in this channel as well. Otherwise, extra excitations are created. The upshot of this discussion is that all pointlike particles in the three-dimensional theory have trivial braiding with one another. This fact is consistent with the fact that any deconfined point particle in three spatial dimensions must be either a fermion or a boson.

As we will see below, however, there is no such restriction for the braiding of a pointlike excitation with a linelike excitation.

The logic for the construction of σ membranes parallels the logic for ψ membranes. A σ membrane in the x - y plane is defined by

$$\hat{\Sigma}_{\hat{z}}^{\sigma}(t, z) := \prod_{J, \tilde{\mathbf{r}}} \hat{\mathcal{O}}_{11, J, \tilde{\mathbf{r}}}^{\sigma}(t, z) \hat{\mathcal{O}}_{22, J, \tilde{\mathbf{r}}}^{\sigma}(t, z), \quad (4.24a)$$

while the membrane operators orthogonal to the x - y plane can be chosen to be

$$\hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon) := \prod_{(\gamma, J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{a}_{\perp}}} \hat{\mathcal{U}}_{\alpha_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}} = \pi} \hat{\mathcal{P}}_{\mathbb{1}} \hat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(t, z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}) \hat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(t, z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}} + \epsilon) \hat{\mathcal{P}}_{\mathbb{1}}, \quad (4.24b)$$

for any $\hat{a} = \hat{x}, \hat{y}$, and where we have defined \hat{a}_{\perp} such that $\hat{x}_{\perp} := \hat{y}$ and $\hat{y}_{\perp} := \hat{x}$. Here, the operator $\hat{\mathcal{P}}_{\mathbb{1}}$ is the counterpart to the projector onto the fusion channel $\sigma \times \sigma = \mathbb{1}$ that appears in Eq. (3.48). The choice of chirality $\mathbf{M} = \mathbf{R}$ is arbitrary, as is the choice of the two sets of base points $\{z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}}\}$ ($\hat{a} = \hat{x}, \hat{y}$) from the noncontractible cycles $\mathcal{P}_{\hat{a}_{\perp}}$, which we abbreviate by $\{z\}$ in the argument of the operator $\hat{\Sigma}_{\hat{a}}^{\sigma}$. The definition (4.24b) of the operator $\hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon)$ is subject to the same caveats as the definition (4.20) of the operator $\hat{\Gamma}_{\hat{z}}^{\sigma}(z_1, \epsilon, t)$. As before, we refer the reader to Footnote 1 and to Appendix C for details.

Using the counterpart to Eq. (3.18) and the definitions (4.24), one can show that the equal-time algebra between any pair of σ -membrane ψ -string operators is mere commutations except for the three anticommuting exceptions

$$\hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon) \hat{\Gamma}_{\hat{a}}^{\psi}(t, z) = -\hat{\Gamma}_{\hat{a}}^{\psi}(t, z) \hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon), \quad (4.25a)$$

$$\hat{\Sigma}_{\hat{z}}^{\sigma}(t, z) \hat{\Gamma}_{\hat{z}}^{\psi}(t) = -\hat{\Gamma}_{\hat{z}}^{\psi}(t) \hat{\Sigma}_{\hat{z}}^{\sigma}(t, z), \quad (4.25b)$$

for $\hat{a} = \hat{x}, \hat{y}$, and for infinitesimal $\epsilon > 0$. Thus, any of the σ -membrane operators can be interpreted as twisting the boundary conditions of a fermion encircling the three-torus along any noncontractible cycle orthogonal to the membrane.

Finally, we also have the algebra between σ membranes and ψ membranes given by

$$\hat{\Sigma}_{\hat{a}}^{\psi}(t) \hat{\Sigma}_{\hat{a}_{\perp}}^{\sigma}(t, \{z\}, \epsilon) = -\hat{\Sigma}_{\hat{a}_{\perp}}^{\sigma}(t, \{z\}, \epsilon) \hat{\Sigma}_{\hat{a}}^{\psi}(t), \quad (4.26a)$$

$$\hat{\Sigma}_{\hat{a}}^{\psi}(t) \hat{\Sigma}_{\hat{z}}^{\sigma}(t, z) = \hat{\Sigma}_{\hat{z}}^{\sigma}(t, z) \hat{\Sigma}_{\hat{a}}^{\psi}(t), \quad (4.26b)$$

$$\hat{\Sigma}_{\hat{z}}^{\psi}(t, z) \hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon) = (-1)^{N_{\hat{y}}} \hat{\Sigma}_{\hat{a}}^{\sigma}(t, \{z\}, \epsilon) \hat{\Sigma}_{\hat{z}}^{\psi}(t, z), \quad (4.26c)$$

	$\hat{\Gamma}_{\hat{x}}^{\psi}$	$\hat{\Gamma}_{\hat{y}}^{\psi}$	$\hat{\Gamma}_{\hat{z}}^{\psi}$	$\hat{\Sigma}_{\hat{x}}^{\psi}$	$\hat{\Sigma}_{\hat{y}}^{\psi}$	$\hat{\Sigma}_{\hat{z}}^{\psi}$	$\hat{\Gamma}_{\hat{x}}^{\sigma}$	$\hat{\Gamma}_{\hat{y}}^{\sigma}$	$\hat{\Gamma}_{\hat{z}}^{\sigma}$	$\hat{\Sigma}_{\hat{x}}^{\sigma}$	$\hat{\Sigma}_{\hat{y}}^{\sigma}$	$\hat{\Sigma}_{\hat{z}}^{\sigma}$
$\hat{\Gamma}_{\hat{x}}^{\sigma}$	+	+	+	-	+	+	+	+	+	✗	+	+
$\hat{\Gamma}_{\hat{y}}^{\sigma}$	+	+	+	+	-	+	+	+	+	+	✗	+
$\hat{\Gamma}_{\hat{z}}^{\sigma}$	+	+	+	+	+	-	+	+	+	+	+	✗
$\hat{\Sigma}_{\hat{x}}^{\sigma}$	-	+	+	+	-	+	✗	+	+	+	✗	✗
$\hat{\Sigma}_{\hat{y}}^{\sigma}$	+	-	+	-	+	+	+	✗	+	✗	+	✗
$\hat{\Sigma}_{\hat{z}}^{\sigma}$	+	+	-	+	+	+	+	+	✗	✗	✗	+

TABLE II. Summary of the algebra of the string and membrane operators (4.27). Entries corresponding to a pair of operators that commute are labeled with a +. Entries corresponding to a pair of operators that anticommute are labeled with a -. Entries corresponding to a pair of operators that neither commute nor anticommute are labeled with a ✗. (Compare with Table I.) The operator algebra contained in the left 6 × 6 subblock of the table is derived in Secs. IV C 1 and IV C 2. The operator algebra contained in the right 6 × 6 subblock of the table is derived in Sec. IV C 3.

for any $\hat{a} = \hat{x}, \hat{y}$ (recall that $\hat{x}_{\perp} = \hat{y}$ and $\hat{y}_{\perp} = \hat{x}$), and for any infinitesimal $\epsilon > 0$. The system-size-dependent integers $N_{\hat{x}}$ and $N_{\hat{y}}$ are the number of wires contained in the path $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, respectively. One can show that $N_{\hat{x}}$ and $N_{\hat{y}}$ are even for paths $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$ that encompass the entire system, so long as the system contains an integer number of unit cells.

3. Topological degeneracy on the three-torus

Using the results of the previous sections, we now derive the topological ground-state degeneracy of the array of quantum wires coupled by the interwire interactions (4.6), for the case of $su(2)_2$ current-current interactions. We assume periodic boundary conditions in x , y , and z ,

so that the array of coupled wires has the topology of a three-torus (\mathbb{T}^3). The logic of our derivation of this lower bound follows closely the logic of the corresponding derivation in the two-dimensional case discussed in Sec. III C. It hinges on the exchange algebra of the following set of nonlocal operators, which is summarized in Table II. There are three nonlocal and *unitary* ψ -string operators (4.12a), (4.12b), and (4.14) for which we use the short-hand notation

$$\hat{\Gamma}_x^\psi, \quad \hat{\Gamma}_y^\psi, \quad \hat{\Gamma}_z^\psi, \quad (4.27a)$$

respectively. There are three nonlocal and *nonunitary* σ -string operators (4.18a), (4.18b), and (4.20) for which we use the short-hand notation

$$\hat{\Gamma}_x^\sigma, \quad \hat{\Gamma}_y^\sigma, \quad \hat{\Gamma}_z^\sigma, \quad (4.27b)$$

respectively. There are three nonlocal and *unitary* ψ -membrane operators (4.15), (4.16a), and (4.16b) for which we use the short-hand notation

$$\hat{\Sigma}_z^\psi, \quad \hat{\Sigma}_y^\psi, \quad \hat{\Sigma}_x^\psi, \quad (4.27c)$$

respectively. There are three nonlocal and *nonunitary* σ -membrane operators, defined in Eqs. (4.24a) and (4.24b), for which we use the short-hand notation

$$\hat{\Sigma}_z^\sigma, \quad \hat{\Sigma}_y^\sigma, \quad \hat{\Sigma}_x^\sigma, \quad (4.27d)$$

respectively. Each of these twelve nonlocal operators commutes with the $su(2)_2$ current-current interaction $\hat{\mathcal{H}}_{\text{bs}}[su(2)_2] \equiv -\hat{\mathcal{L}}_{\text{bs}}[su(2)_2]$ that couples neighboring wires [recall Eq. (4.2a)], except for the three operators $\hat{\Gamma}_z^\sigma$, $\hat{\Sigma}_x^\sigma$, and $\hat{\Sigma}_y^\sigma$, defined in Eqs. (4.20) and (4.24b), respectively. These three operators are regularized by the parameter ϵ , and therefore must be treated in a manner similar to the operator $\hat{\Gamma}_{2,y'}^\sigma(z_1, \epsilon)$ in the 2D case. Nevertheless, an analysis along the lines of the one presented in Appendix C for the 2D case reveals that these three ϵ -regularized operators can be used to define states in the ground-state manifold of the interaction $\hat{\mathcal{H}}_{\text{bs}}[su(2)_2]$. We will elaborate on this statement below.

The derivation of the topological degeneracy begins by

observing that the ψ -string and ψ -membrane operators appearing in Eqs. (4.27a) and (4.27c) all commute with one another. Thus, we can choose a many-body ground state

$$|\Omega\rangle \equiv |\mathbb{1}\rangle \quad (4.28)$$

that is a simultaneous eigenstate of all ψ -string and ψ -membrane operators, namely

$$\hat{\Gamma}_z^\psi |\mathbb{1}\rangle = \omega_z^\Gamma |\mathbb{1}\rangle, \quad \hat{\Gamma}_y^\psi |\mathbb{1}\rangle = \omega_y^\Gamma |\mathbb{1}\rangle, \quad \hat{\Gamma}_x^\psi |\mathbb{1}\rangle = \omega_x^\Gamma |\mathbb{1}\rangle, \quad (4.29a)$$

on the one hand, and

$$\hat{\Sigma}_z^\psi |\mathbb{1}\rangle = \omega_z^\Sigma |\mathbb{1}\rangle, \quad \hat{\Sigma}_x^\psi |\mathbb{1}\rangle = \omega_x^\Sigma |\mathbb{1}\rangle, \quad \hat{\Sigma}_y^\psi |\mathbb{1}\rangle = \omega_y^\Sigma |\mathbb{1}\rangle, \quad (4.29b)$$

on the other hand, must hold for the nonvanishing eigenvalues

$$\omega_z^\Gamma, \omega_y^\Gamma, \omega_x^\Gamma, \omega_z^\Sigma, \omega_y^\Sigma, \omega_x^\Sigma \in U(1). \quad (4.29c)$$

Not all choices of $|\Omega\rangle$ are equivalent. Similarly to the argument presented in Appendix C for the 2D case, depending on the topological sector in which the state $|\Omega\rangle$ resides, it is possible for the state created by acting upon $|\Omega\rangle$ with certain combinations of the nonlocal, nonunitary operators $\hat{\Gamma}_{x,y,z}^\sigma$ and $\hat{\Sigma}_{x,y,z}^\sigma$ to have norm zero or infinity. In other words, not all combinations of the eigenvalues (4.29c) label states in the ground-state manifold.

Next, we define a set of many-body states obtained by acting on the state $|\mathbb{1}\rangle$ with the σ -string and σ -membrane operators from Eqs. (4.27b) and (4.27d), respectively. There are

$$4^3 = 64 \quad (4.30)$$

states, since for any choice of a noncontractible cycle of the three-torus (\hat{x} , \hat{y} , or \hat{z}), there are four nonlocal operators we can apply to the state $|\Omega\rangle$. For example, fixing the noncontractible cycle \hat{x} , we can insert the identity operator $\mathbb{1}$, the σ -string operator $\hat{\Gamma}_x^\sigma$, the σ -membrane operator $\hat{\Sigma}_x^\sigma$, or the product $\hat{\Gamma}_x^\sigma \hat{\Sigma}_x^\sigma$. Alternatively, we can label all $2^6 = 64$ states according to the rule

$$\left| \left(\hat{\Sigma}_x^\sigma \right)^{\sigma_x^\Sigma} \left(\hat{\Sigma}_y^\sigma \right)^{\sigma_y^\Sigma} \left(\hat{\Sigma}_z^\sigma \right)^{\sigma_z^\Sigma} \left(\hat{\Gamma}_x^\sigma \right)^{\sigma_x^\Gamma} \left(\hat{\Gamma}_y^\sigma \right)^{\sigma_y^\Gamma} \left(\hat{\Gamma}_z^\sigma \right)^{\sigma_z^\Gamma} \right\rangle := \left(\hat{\Sigma}_x^\sigma \right)^{\sigma_x^\Sigma} \left(\hat{\Sigma}_y^\sigma \right)^{\sigma_y^\Sigma} \left(\hat{\Sigma}_z^\sigma \right)^{\sigma_z^\Sigma} \left(\hat{\Gamma}_x^\sigma \right)^{\sigma_x^\Gamma} \left(\hat{\Gamma}_y^\sigma \right)^{\sigma_y^\Gamma} \left(\hat{\Gamma}_z^\sigma \right)^{\sigma_z^\Gamma} |\Omega\rangle, \quad (4.31)$$

where $\sigma_x^\Gamma, \sigma_y^\Gamma, \sigma_z^\Gamma, \sigma_x^\Sigma, \sigma_y^\Sigma, \sigma_z^\Sigma = 0, 1$. Any of the above states involving one or more of the ϵ -regularized operators $\hat{\Gamma}_z^\sigma$, $\hat{\Sigma}_x^\sigma$, and $\hat{\Sigma}_y^\sigma$ carries an implicit limit $\epsilon \rightarrow 0$. As in the 2D case (see Footnote 1 and Appendix C), this limit should be taken after forming the product of the relevant string and/or membrane operators.

Not all states of the form (4.31) belong to the ground state manifold, as we are going to show explicitly. The counting based on Eq. (4.30) is “naive” because it is based purely on the number of noncontractible cycles of the manifold on which the theory is defined, and on the number of string or membrane operators that can act

State	Eigenvalues						
$ \mathbb{1}\rangle$	(+, +, +, +, +, +)	$ \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, +, +, -, +, -)	$ \hat{\Sigma}_z^\sigma\rangle$	(+, +, -, +, +, +)	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	(-, +, +, +, -, -)
$ \hat{\Gamma}_x^\sigma\rangle$	(+, +, +, -, +, +)	$ \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, +, +, +, -, -)	$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma\rangle$	(+, -, +, +, +, +)	$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, -, +, -, +, -)
$ \hat{\Gamma}_y^\sigma\rangle$	(+, +, +, +, -, +)	$ \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, +, +, -, -, -)	$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma\rangle$	(+, +, -, -, +, +)	$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, +, -, -, -, +)
$ \hat{\Gamma}_z^\sigma\rangle$	(+, +, +, +, +, -)	$ \hat{\Sigma}_x^\sigma\rangle$	(-, +, +, +, -, +)	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	(-, +, +, +, +, +)	$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	(+, -, +, +, +, -)
$ \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	(+, +, +, -, -, +)	$ \hat{\Sigma}_y^\sigma\rangle$	(+, -, +, -, +, +)	$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma\rangle$	(+, +, -, +, -, +)	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	(-, +, +, +, +, -)

TABLE III. The 20 orthogonal states of the form (4.31) that span the ground-state manifold of the $su(2)_2$ coupled-wire theory in $(3+1)$ -dimensional spacetime, as well as the eigenvalues of these states under the ψ -string and membrane operators. The states are labeled according to the notation $|\hat{\mathcal{O}}\rangle = \hat{\mathcal{O}}|\Omega\rangle$. The 6-tuple of signs \pm indicating the eigenvalues of a state $|\hat{\mathcal{O}}\rangle$ is obtained by evaluating the list of matrix elements $\langle\hat{\mathcal{O}}|(\hat{\Gamma}_x^\psi, \hat{\Gamma}_y^\psi, \hat{\Gamma}_z^\psi, \hat{\Sigma}_x^\psi, \hat{\Sigma}_y^\psi, \hat{\Sigma}_z^\psi)|\hat{\mathcal{O}}\rangle$ and dividing each element in the list by its magnitude.

along each noncontractible cycle. In the following, we are going to show that a majority of the states (4.31) must be excluded from the ground-state manifold, on grounds similar to the reason for which we had to exclude the “extra” state $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle$ that appeared in the two-dimensional example discussed in Sec. III C 2. In the end, there will be a total of

$$D_{\mathbb{T}^3} := 20 \quad (4.32)$$

states that survive projection into the ground-state manifold. These states are listed, along with their eigenvalues under the ψ string and membrane operators, in Table III. We emphasize that all of these 20 ground states are mutually orthogonal, as they are simultaneous eigenstates of the unitary ψ string and membrane operators with different eigenvalues.

The 20 states in Table III have in common the fact that they are created by acting on the state $|\Omega\rangle$ with a product of commuting operators (these correspond to entries marked with a + in the right 6×6 block of Table II).² Conversely, the $64 - 20 = 44$ excluded states not appearing in Table III are created by acting on the state $|\Omega\rangle$

with a product of noncommuting operators (these correspond to entries marked with a \times in the right 6×6 block of Table II). In the two-dimensional case studied in Sec. III C 2, it was precisely the noncommutativity of the string operators $\hat{\Gamma}_1^\sigma$ and $\hat{\Gamma}_2^\sigma$ that led to the exclusion of the state $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle$ from the ground-state manifold. In the proof below, we follow this “algebraic” approach, identifying all noncommuting pairs of σ -string and σ -membrane operators from Eqs. (4.27b) and (4.27d), respectively. The noncommuting σ -string-membrane pairs consist of strings and membranes that are perpendicular to one another, intersecting in a point. The noncommuting σ -membrane-membrane pairs consist of perpendicular membranes, whose intersection is a line. Whenever a noncommuting pair of operators acts on one of the states in Table III, we will show that the resulting state must be excluded from the ground-state manifold. (A complementary “analytic” proof that these states must be excluded, along the lines of Appendix C, could also be undertaken, but we do not do this here.)

We now proceed with the proof. Of key importance is the projection operator

$$\hat{\mathcal{P}}_{\text{GSM}} := \mathcal{N}_{\mathbb{1}}^{-1} |\mathbb{1}\rangle \langle \mathbb{1}| + \mathcal{N}_{\hat{\Gamma}_x^\sigma}^{-1} |\hat{\Gamma}_x^\sigma\rangle \langle \hat{\Gamma}_x^\sigma| + \mathcal{N}_{\hat{\Gamma}_y^\sigma}^{-1} |\hat{\Gamma}_y^\sigma\rangle \langle \hat{\Gamma}_y^\sigma| + \mathcal{N}_{\hat{\Gamma}_z^\sigma}^{-1} |\hat{\Gamma}_z^\sigma\rangle \langle \hat{\Gamma}_z^\sigma| + \dots \quad (4.33)$$

onto the ground state manifold, c.f. Eq. (3.58). Here, $\mathcal{N}_{\hat{\mathcal{O}}}$ is the squared norm of the state $|\hat{\mathcal{O}}\rangle$, and \dots is a sum over the remaining elements of the orthonormal basis of the ground state manifold, including the states listed in Table III. By construction, $\hat{\mathcal{P}}_{\text{GSM}}$ leaves any state in Table III

invariant, and, being a projector, satisfies

$$\hat{\mathcal{P}}_{\text{GSM}}^2 = \hat{\mathcal{P}}_{\text{GSM}}. \quad (4.34)$$

Furthermore, the projector $\hat{\mathcal{P}}_{\text{GSM}}$ satisfies

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\mathcal{O}} \hat{\mathcal{P}}_{\text{GSM}} = 0 \quad (4.35)$$

for any operator $\hat{\mathcal{O}}$ whose action on any of the states in Table III produces an excited state. To prove that the 44 states in question must be excluded from the ground state manifold, we will show for two particular classes of operators $\hat{\mathcal{O}}$ that Eq. (4.35) holds. This will turn out to

² Any of the states in Table III involving one or more of the ϵ -regularized operators $\hat{\Gamma}_z^\sigma$, $\hat{\Sigma}_x^\sigma$, and $\hat{\Sigma}_y^\sigma$ can be shown to be a ground state of the interaction $\hat{\mathcal{H}}_{\text{bs}}[su(2)_2]$ by an argument along the lines of Appendix C.

State	Eigenvalues				
$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma\rangle$	$(-, +, +, -, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, +, -, -, +, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, +, +, -, +, +)$
$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(+, -, +, -, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, +, +, -, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, +, -, -, -)$
$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, +, -, +, +, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(+, -, -, +, -, +)$	$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(+, -, +, +, -, +)$
$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, +, -, +, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, +, -, +, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, +, -, -, -)$
$ \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, +, +, -, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, -, +, +, -)$	$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, +, -, -, +, -)$
$ \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, +, -, -, -, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, -, +, +, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, +, -, +, -, -)$
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma\rangle$	$(-, -, +, +, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma\rangle$	$(-, +, -, -, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, -, +, -, +, +)$
$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(+, -, -, -, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, -, +, -, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, -, -, +, -)$

TABLE IV. The 24 orthogonal states of the form (4.31) that are excluded based on Eqs. (4.37) and (4.38), as well as the eigenvalues of these states under the ψ -string and membrane operators. The notation for states and eigenvalues is as in Table III.

be sufficient to exclude the offending states.

The first class of operators arises when we consider products of perpendicular σ -strings and σ -membranes. This includes the three operators

$$\hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma, \quad \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma, \quad \hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma, \quad (4.36a)$$

as well as products of the form

$$\hat{\Sigma}_a^\sigma \hat{\Gamma}_a^\sigma \hat{\mathcal{O}}, \quad (4.36b)$$

for $\hat{a} = \hat{x}, \hat{y}, \hat{z}$ and any string or membrane operator \mathcal{O} that commutes with $\hat{\Sigma}_a^\sigma$ and $\hat{\Gamma}_a^\sigma$. The operators in Eq. (4.36a) share the common trait that the domain of intersection of the string and membrane operators contains three $\hat{\sigma}$ operators per chiral channel. The exchange algebra relevant to this case was computed in Sec. III C 2. By small variations on the calculation presented in Eqs. (3.66) and (3.67), one verifies the relations

$$\hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma = e^{+i\frac{\pi}{4}} \hat{\Gamma}_z^\sigma \hat{\Sigma}_z^\sigma, \quad (4.37a)$$

$$\hat{\Gamma}_x^\sigma \hat{\Sigma}_x^\sigma = e^{+i\frac{\pi}{4}} \hat{\Sigma}_x^\sigma \hat{\Gamma}_x^\sigma, \quad (4.37b)$$

$$\hat{\Gamma}_y^\sigma \hat{\Sigma}_y^\sigma = e^{+i\frac{\pi}{4}} \hat{\Sigma}_y^\sigma \hat{\Gamma}_y^\sigma, \quad (4.37c)$$

where the operators $\hat{\Gamma}_i^\sigma$ and $\hat{\Sigma}_i^\sigma$ are defined in the same way as the operator $\hat{\Gamma}_2^\sigma$ appearing in Eq. (3.67), i.e., one replaces any appearance of $\hat{\mathcal{P}}_1 \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z) \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z + \epsilon) \hat{\mathcal{P}}_1$ in the *intersection* of the string/membrane pair with $\hat{\mathcal{P}}_\psi \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z) \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z + \epsilon) \hat{\mathcal{P}}_\psi$, and leaves all other appearances of $\hat{\mathcal{P}}_1 \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z) \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z + \epsilon) \hat{\mathcal{P}}_1$ unchanged. Each of Eqs. (4.37) carries an implicit limit $\epsilon \rightarrow 0$. Whether the tilde appears on the string or the membrane operator above depends on which operator contains a product of two $\hat{\sigma}$ operators in the same wire. By direct analogy with the two-dimensional case, we have

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\Gamma}_z^\sigma \hat{\mathcal{P}}_{\text{GSM}} = 0, \quad (4.38a)$$

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\Sigma}_x^\sigma \hat{\mathcal{P}}_{\text{GSM}} = 0, \quad (4.38b)$$

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\Sigma}_y^\sigma \hat{\mathcal{P}}_{\text{GSM}} = 0, \quad (4.38c)$$

in the limit $\epsilon \rightarrow 0$. Combining Eqs. (4.37) and (4.38), one can show, in direct analogy with Eq. (3.71) in the two-dimensional case, that any state created by acting with any operator of the 3×8 operators of the form (4.36) on $|1\rangle$ must be excluded from the ground-state manifold. This is sufficient to eliminate 24 of the 44 “extra” states of the form (4.31) that are not in the Table III. These eliminated states are listed in Table IV.

The second class of operators arises when we consider products of two perpendicular membranes, e.g.,

$$\hat{\Sigma}_z^\sigma \hat{\Sigma}_x^\sigma, \quad \hat{\Sigma}_z^\sigma \hat{\Sigma}_y^\sigma, \quad \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma, \quad (4.39a)$$

as well as products of the form

$$\hat{\Sigma}_a^\sigma \hat{\Sigma}_b^\sigma \hat{\mathcal{O}}, \quad (4.39b)$$

where $\hat{a}, \hat{b} = \hat{x}, \hat{y}, \hat{z}$, for $\hat{a} \neq \hat{b}$ and any operator $\hat{\mathcal{O}}$ that commutes with $\hat{\Sigma}_a^\sigma$ and $\hat{\Sigma}_b^\sigma$. It turns out that the operators $\hat{\Sigma}_z^\sigma \hat{\Sigma}_x^\sigma$ and $\hat{\Sigma}_z^\sigma \hat{\Sigma}_y^\sigma$ can also be handled using minor variations on the calculation presented in Eqs. (3.66) and (3.67) of Sec. III C 2. Specifically, one can show the relations

$$\hat{\Sigma}_z^\sigma \hat{\Sigma}_a^\sigma = e^{+iN_{a\perp}\frac{\pi}{4}} \hat{\Sigma}_a^{\sigma'} \hat{\Sigma}_z^\sigma, \quad (4.40a)$$

in the limit $\epsilon \rightarrow 0$, for $\hat{a} = \hat{x}, \hat{y}$, and for $\hat{x}_\perp = \hat{y}$ and $\hat{y}_\perp = \hat{x}$. The system-size-dependent integers $N_{\hat{x}}$ and $N_{\hat{y}}$ are the number of wires contained in the path $\mathcal{P}_{\hat{x}}$ and $\mathcal{P}_{\hat{y}}$, respectively. The operators $\hat{\Sigma}_x^{\sigma'}$ and $\hat{\Sigma}_y^{\sigma'}$ are again defined by direct analogy with the operator $\hat{\Gamma}_2^\sigma$ appearing in Eq. (3.67), i.e., by replacing *any* appearance of $\hat{\mathcal{P}}_1 \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z) \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z + \epsilon) \hat{\mathcal{P}}_1$ with $\hat{\mathcal{P}}_\psi \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z) \hat{\sigma}_{\gamma, M, J, \tilde{r}}(z + \epsilon) \hat{\mathcal{P}}_\psi$. The reason for which we use the primes here is to distinguish these opera-

State	Eigenvalues		
$ \widehat{\Sigma}_x^\sigma \widehat{\Sigma}_z^\sigma\rangle$	$(-, +, -, +, -, +)$	$ \widehat{\Sigma}_y^\sigma \widehat{\Sigma}_z^\sigma\rangle$	$(+, -, -, -, +, +)$
$ \widehat{\Sigma}_x^\sigma \widehat{\Sigma}_y^\sigma\rangle$	$(-, -, +, -, -, +)$	$ \widehat{\Sigma}_x^\sigma \widehat{\Sigma}_z^\sigma \widehat{\Gamma}_y^\sigma\rangle$	$(-, +, -, +, +, +)$
$ \widehat{\Sigma}_y^\sigma \widehat{\Sigma}_z^\sigma \widehat{\Gamma}_x^\sigma\rangle$	$(+, -, -, +, +, +)$	$ \widehat{\Sigma}_x^\sigma \widehat{\Sigma}_y^\sigma \widehat{\Gamma}_z^\sigma\rangle$	$(-, -, +, -, -, -)$
$ \widehat{\Sigma}_x^\sigma \widehat{\Sigma}_y^\sigma \widehat{\Sigma}_z^\sigma\rangle$	$(-, -, -, -, -, +)$		

TABLE V. The seven orthogonal states of the form (4.31) that are excluded based on Eqs. (4.40), (4.44), (4.41), and (4.46), as well as the eigenvalues of these states under the ψ -string and membrane operators. The notation for states and eigenvalues is as in Table III.

tors from $\widehat{\Sigma}_x^\sigma$ and $\widehat{\Sigma}_y^\sigma$, where only *some* of the appearances of $\widehat{\mathcal{P}}_\perp \widehat{\sigma}_{\gamma, \mathbf{M}, J, \tilde{\mathbf{r}}}(z) \widehat{\sigma}_{\gamma, \mathbf{M}, J, \tilde{\mathbf{r}}}(z + \epsilon) \widehat{\mathcal{P}}_\perp$ are replaced by $\widehat{\mathcal{P}}_\psi \widehat{\sigma}_{\gamma, \mathbf{M}, J, \tilde{\mathbf{r}}}(z) \widehat{\sigma}_{\gamma, \mathbf{M}, J, \tilde{\mathbf{r}}}(z + \epsilon) \widehat{\mathcal{P}}_\psi$. Regardless of these slight differences in definition, the operators $\widehat{\Sigma}_x^{\sigma'}$ and $\widehat{\Sigma}_y^{\sigma'}$ create excited states when acting on the vacuum $|\Omega\rangle$. Consequently, we have

$$\widehat{\mathcal{P}}_{\text{GSM}} \widehat{\Sigma}_a^{\sigma'} \widehat{\mathcal{P}}_{\text{GSM}} = 0, \quad (4.41a)$$

in the limit $\epsilon \rightarrow 0$ for $\hat{a} = \hat{x}, \hat{y}$. The operator $\widehat{\Sigma}_x^\sigma \widehat{\Sigma}_y^\sigma$, which involves four $\widehat{\sigma}$ operators per chiral channel contained in the intersection of the two membranes, can be treated similarly. The exchange algebra of the membrane operators $\widehat{\Sigma}_x^\sigma$ and $\widehat{\Sigma}_y^\sigma$ can be determined by considering the diagram

$$, \quad (4.42)$$

which obeys

$$. \quad (4.43)$$

The algebraic interpretation of this diagrammatic statement is

$$\widehat{\Sigma}_x^\sigma \widehat{\Sigma}_y^\sigma = e^{+i \frac{3\pi}{4}} \widehat{\Sigma}_y^\sigma \widehat{\Sigma}_x^\sigma, \quad (4.44)$$

in the limit $\epsilon \rightarrow 0$, where the operators $\widehat{\Sigma}_y^\sigma$ and $\widehat{\Sigma}_x^\sigma$ also appear in Eqs. (4.37). Explicitly, we have [compare Eq. (4.24b)]

$$\widehat{\Sigma}_a^\sigma := \left(\prod_{(\gamma, J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{a}_\perp} \setminus (\mathcal{P}_{\hat{a}_\perp} \cap \mathcal{P}_{\hat{a}})} \widehat{U}_{\alpha_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}} = \pi} \widehat{\mathcal{P}}_\perp \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}) \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}} + \epsilon) \widehat{\mathcal{P}}_\perp \right) \quad (4.45a)$$

$$\times \prod_{(\gamma, J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{a}_\perp} \cap \mathcal{P}_{\hat{a}}} \widehat{U}_{\alpha_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}} = \pi} \widehat{\mathcal{P}}_\psi \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}) \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}} + \epsilon) \widehat{\mathcal{P}}_\psi$$

$$\sim \left(\prod_{(\gamma, J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{a}_\perp} \setminus (\mathcal{P}_{\hat{a}_\perp} \cap \mathcal{P}_{\hat{a}})} \widehat{U}_{\alpha_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}} = \pi} \widehat{\mathcal{P}}_\perp \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}) \widehat{\sigma}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}} + \epsilon) \widehat{\mathcal{P}}_\perp \right) \quad (4.45b)$$

$$\times \prod_{(\gamma, J, \tilde{\mathbf{r}}) \in \mathcal{P}_{\hat{a}_\perp} \cap \mathcal{P}_{\hat{a}}} \widehat{U}_{\alpha_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}} = \pi} \widehat{\psi}_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}}(z_{\gamma, \mathbf{R}, J, \tilde{\mathbf{r}}; \hat{a}) + \dots,$$

for $\hat{a} = \hat{x}, \hat{y}$ and for infinitesimal $\epsilon > 0$ (recall that $\hat{x}_\perp = \hat{y}$ and $\hat{y}_\perp = \hat{x}$), where in the second line

we have performed the OPE in the wires where the two membranes intersect. Thus, the chiral channel in which the two membranes intersect [see Fig. 7(b)] contains a pair of fermion excitations due to the product $\hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{y}})\hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{x}})$. Note that the points $z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{y}}$ and $z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{x}}$ are arbitrary, and thus that the fermion excitations $\hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{y}})$ and $\hat{\psi}_{\gamma,R,J,\tilde{\mathbf{r}}}(z_{\gamma,R,J,\tilde{\mathbf{r}};\hat{x}})$ can be separated by arbitrarily large distances along the z -axis. Consequently, we have

$$\hat{\mathcal{P}}_{\text{GSM}} \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\mathcal{P}}_{\text{GSM}} = e^{+i\frac{3\pi}{4}} \hat{\mathcal{P}}_{\text{GSM}} \hat{\Sigma}_y^\sigma \hat{\Sigma}_x^\sigma \hat{\mathcal{P}}_{\text{GSM}} = 0, \quad (4.46)$$

in the limit $\epsilon \rightarrow 0$. Using Eqs. (4.40), (4.44), (4.41), and (4.46), one can show that the seven states listed in Table V are eliminated from the ground-state manifold.

Finally, the $44 - 24 - 7 = 13$ remaining “extra” states of the form (4.31) can also be eliminated using appropriate combinations of Eqs. (4.37), (4.38), (4.40), (4.44), (4.41), and (4.46). These states are listed in Table VI. In all cases, the reason for elimination is the same: each state is created by acting on one of the states in Table III with an operator that creates an excess of fermion excitations.

To summarize, we have shown that of the $2^6 = 64$ states labeled by the eigenvalues of the ψ -string or ψ -membrane operators, only the 20 listed in Table III truly reside in the ground-state manifold once the exchange algebra of the σ -string and σ -membrane operators is taken into account. This exchange algebra is highly non-trivial, because reordering a product of σ -string and/or σ -membrane operators not only produces simple multiplicative phase factors, but enacts nontrivial unitary operations within the space spanned by the operator products. As in the two-dimensional case discussed in Sec. III C 2, this reduction of the number of states in the ground-state manifold from the naive value lies at the heart of the distinction between Abelian and non-Abelian topological states of matter.

V. SURFACE THEORY

Let us now remove the periodic boundary conditions imposed in the previous section and replace them with boundary conditions that are open along the x -direction and periodic along the y -direction. The bulk of the (3+1)-dimensional coupled-wire theory then possesses the “time-reversal” symmetry $\mathcal{T}_{\text{eff},\hat{y}}$ defined in Eq. (4.3b). In this case, the inter-wire interaction (4.2a) is then not sufficient to gap all gapless $su(2)_k$ modes; there remain gapless modes that are confined to the surfaces at $x = 0$ and $x = L_x$. In this section, we investigate the fate of these gapless surface modes when they are coupled by marginally relevant current-current interactions. From now on, we shall only consider the surface at $x = 0$.

The surface at $x = 0$ supports gapless modes that can be represented by a quadratic form for the currents that

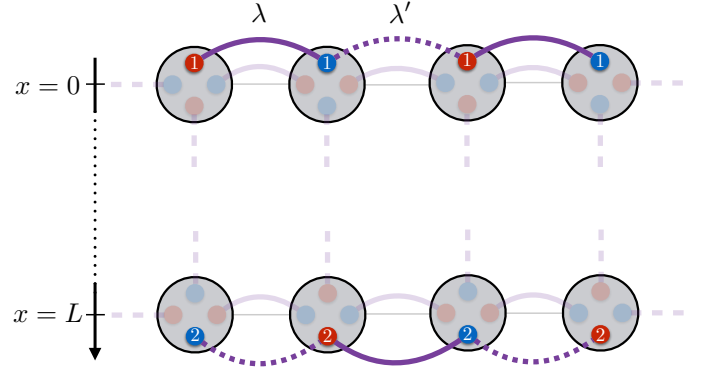


FIG. 9. (Color online) Surface of the coupled-wire theory with the inter-wire interactions (4.2a) for the case of mixed periodic and open boundary conditions. The graphical notation of Fig. 6 is used here. When open boundary conditions are imposed along the x -direction, while periodic boundary conditions are imposed along the y - and z -directions, the interactions (4.2a) leave gapless modes on the surfaces at $x = 0$ and $x = L_x$. The purple bonds connecting chiral modes on the surface represent the interactions (5.2a) in the $SU(2)$ -symmetric limit (5.2b). When $\lambda = \lambda'$, the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$ is present.

generate the copy $\gamma = 1$ of the $su(2)_k$ affine Lie algebra with the M-moving currents $J_{\gamma=1,M,y}^a$ (the label $\gamma = 2$ applies to the surface $x = L_x$, see Fig. 9). From now on, we will drop the explicit reference to the label $\gamma = 1$. Hence, the Hamiltonian density for the gapless modes on the surface at $x = 0$ is the linear combination

$$\mathcal{H}_{x=0} := 2\pi (T_{L,A,x=0}[su(2)_k] + T_{R,B,x=0}[su(2)_k]), \quad (5.1a)$$

where

$$T_{M,J,x=0}[su(2)_k] = \frac{1}{2+k} \sum_{y=0}^{L_y} \sum_{a=1}^3 J_{M,J,y}^a J_{M,J,y}^a \quad (5.1b)$$

is the energy-momentum tensor for the M-moving mode on sublattice J . Here, the priming of the sum over y indicates that only *even* wires are to be summed over. Summing only over even y and over $J = A, B$ accounts for the “dangling” gapless modes on the $x = 0$ surface that do not couple to any neighbors via the couplings depicted in Fig. 6 when open boundary conditions are imposed in the x -direction. We assume that L_y is odd, so that the total number of wires (i.e., $L_y + 1$) is even.

The surface theory at $x = 0$, whose energy-momentum tensor has the chiral components (5.1), can be viewed as a conformal field theory in (1+1)-dimensional space-time with an extensive central charge. We would like to decrease this central charge to a finite number in the thermodynamic limit ($L_y \rightarrow \infty$). To this end, we perturb

State	Eigenvalues				
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma\rangle$	$(-, -, -, +, -, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, -, -, -, +, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, -, -, -, -)$
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, -, +, +, +, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, -, -, -, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, -, -, -, -)$
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, +, +, +, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, +, -, -, +, -)$	$ \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(+, -, -, +, -, -)$
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma\rangle$	$(-, -, -, +, +, +)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, -, +, -, -)$	$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, -, -, +, -)$
$ \hat{\Sigma}_x^\sigma \hat{\Sigma}_y^\sigma \hat{\Sigma}_z^\sigma \hat{\Gamma}_x^\sigma \hat{\Gamma}_y^\sigma \hat{\Gamma}_z^\sigma\rangle$	$(-, -, -, +, +, -)$				

TABLE VI. The 13 orthogonal states of the form (4.31) that are excluded based on appropriate combinations of Eqs. (4.37), (4.38), (4.40), (4.44), (4.41), and (4.46), as well as the eigenvalues of these states under the ψ -string and membrane operators. The notation for states and eigenvalues is as in Table III.

the gapless theory (5.1) with the interactions

$$\mathcal{L}_{\text{bs}, x=0} = - \sum_{y=0}^{L_y} \sum_{a=1}^3 (\lambda^a J_{\text{R}, y}^a J_{\text{L}, y+1}^a + \lambda'^a J_{\text{L}, y+1}^a J_{\text{R}, y+2}^a). \quad (5.2a)$$

To investigate the nature of the surface more closely, we allow the possibility that this surface interaction breaks explicitly the $SU(2)$ symmetry. The choices

$$\lambda \equiv \lambda^a, \quad \lambda' \equiv \lambda'^a, \quad a = 1, 2, 3, \quad (5.2b)$$

restore the explicit $SU(2)$ symmetry of the bulk. These couplings are depicted in Fig. 9.

For the isotropic point (5.2b), it is readily shown that there are two gapped phases, one for $\lambda > \lambda' \geq 0$ and one for $0 \leq \lambda < \lambda'$, that are related to one another by the “time-reversal” symmetry $\mathcal{T}_{\text{eff}, \hat{y}}$ defined in Eq. (4.3b). Indeed, when $\lambda' = 0$ and $\lambda > 0$ (or vice versa) the interactions (5.2a) are marginally relevant, flowing to strong coupling and opening a gap, as they do in the bulk. Furthermore, if we define a “magnetic” domain wall at $y = 0$ by allowing λ and λ' to acquire the y -dependent profiles

$$\lambda_y = \begin{cases} \lambda_{-\infty} > 0, & \text{if } y < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (5.3a)$$

and

$$\lambda'_y = \begin{cases} 0, & \text{if } y < 0, \\ \lambda'_{-\infty} > 0, & \text{otherwise,} \end{cases} \quad (5.3b)$$

respectively, one finds that a single chiral $su(2)_k$ current is localized at the domain wall. This is reminiscent of the surface physics of the usual 3D TI, where a domain wall between different TRS-broken regions on the surface binds a chiral $u(1)_1$ current (i.e., a chiral Dirac fermion mode). In the present setting, the role of TRS is played by the non-onsite symmetry operation $\mathcal{T}_{\text{eff}, \hat{y}}$. This non-onsite implementation of TRS is an artifact of the coupled-wire construction; this is also the case, for example, in the coupled-wire constructions presented in Refs. [72] and [73]. However, we expect that an analogous family of phases with an onsite implementation of TRS exists.

The remainder of this section is devoted to elucidating the nature of the surface theory for the $\mathcal{T}_{\text{eff}, \hat{y}}$ -symmetric (but not necessarily $SU(2)$ -symmetric) case

$$\lambda^a = \lambda'^a, \quad a = 1, 2, 3. \quad (5.4)$$

First, we present a one-loop renormalization group (RG) analysis, valid for small magnitudes of λ^a and λ'^a with $a = 1, 2, 3$. This analysis sheds light on the phase diagram of the surface, particularly on the response of the surface theory to $SU(2)$ -breaking perturbations. Next, we shall present a mean-field analysis of the surface theory for the case $k = 2$. This analysis demonstrates that the point $\lambda = \lambda' > 0$, a strongly interacting quantum field theory when expressed in terms of the original fermionic modes, is a continuous critical point that can be described by two noninteracting modes. The first mode is a gapless complex-valued fermion realizing a single Dirac cone in the low-energy limit. The second mode is a gapless real-valued fermion realizing a single Majorana cone in the low-energy limit. These low-energy modes are surface states of the $su(2)_2$ coupled-wire theory that are protected by the symmetry $\mathcal{T}_{\text{eff}, \hat{y}}$.

A. One-loop RG analysis

We now perform a one-loop RG analysis of the surface interaction (5.2a) in the presence of both λ^a and λ'^a with $a = 1, 2, 3$ under the assumption that these couplings are small. Hence, the bare surface interaction (5.2a) is a small perturbation to the critical surface theory with the energy-momentum tensor (5.1). The RG calculation itself is standard, and makes use of the current-current OPEs (3.3) (see, e.g., [92]). The resulting RG equations describing the flow of the couplings λ^a and λ'^a as functions of the cutoff length scale ℓ are

$$\frac{d\lambda^a}{d\ell} = +2\pi \lambda^b \lambda^c, \quad (5.5a)$$

$$\frac{d\lambda'^a}{d\ell} = +2\pi \lambda'^b \lambda'^c, \quad (5.5b)$$

for $1 \leq a < b < c \leq 3$ and cyclic permutations thereof. Note that at the $SU(2)$ -symmetric point (5.2b), the RG

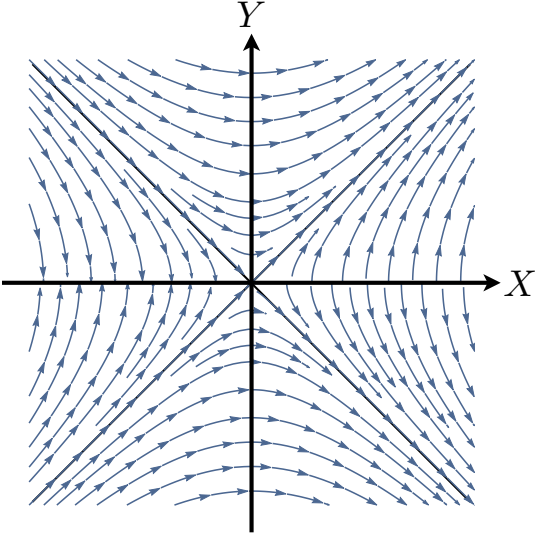


FIG. 10. One-loop renormalization group flows (5.6b).

flows (5.5) indicate that the couplings λ and λ' are marginally relevant, as is the case in the bulk.

The one-loop renormalization-group flows for the triplet λ^1 , λ^2 , and λ^3 have decoupled from those of the triplet λ'^1 , λ'^2 , and λ'^3 ; in fact, they are identical at the one-loop level. We expect this to be true to all orders in perturbation theory, since the interaction (5.2a) is form-invariant under composing the transformation $\mathcal{T}_{\text{eff},\hat{y}}$ with the interchange of λ^a and λ'^a . Thus, if the flow starts from an initial condition such that $\lambda^a = \lambda'^a$ for all $a = 1, 2, 3$ (as must be the case for a surface that does not explicitly break the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$), the asymmetry $\lambda - \lambda' = 0$ for all $\ell > 0$. If initial conditions are chosen such that $\lambda^a, \lambda'^a > 0$ for all (or even, as we shall see below, for only some) $a = 1, 2, 3$, then all couplings λ^a and λ'^a flow to infinity. For initial conditions that do not respect the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$, then we expect this strong-coupling fixed point to break the symmetry as well.

We now focus on the $\mathcal{T}_{\text{eff},\hat{y}}$ -symmetric case ($\lambda^a = \lambda'^a$ for all $a = 1, 2, 3$) and investigate the fate of $SU(2)$ symmetry under the RG flows (5.5). By analyzing vector field plots for the differential equations (5.5), one can convince oneself that the strong-coupling fixed point reached from initial conditions $\lambda^a > 0$ for $a = 1, 2, 3$ is in fact $SU(2)$ -symmetric. Thus, even if the initial conditions do not satisfy the conditions (5.2b), the strong-coupling fixed point does. We will illustrate this below for the $U(1)$ -symmetric case, which is easier to visualize as the phase diagram is then two- rather than three-dimensional.

Let us analyze in greater detail the $\mathcal{T}_{\text{eff},\hat{y}}$ - and $U(1)$ -symmetric case

$$\lambda^a = \lambda'^a, \quad \lambda^1 = \lambda^2 \equiv \lambda_{\perp}, \quad \lambda^3 \equiv \lambda_{\parallel}, \quad (5.6a)$$

for which the one-loop renormalization-group flows (5.5)

simplify to

$$\frac{dX}{d\ell} = +2\pi Y^2, \quad \frac{dY}{d\ell} = +2\pi X Y, \quad (5.6b)$$

where either $(X, Y) = (\lambda_{\parallel}, \lambda_{\perp})$ or $(X, Y) = (\lambda'_{\parallel}, \lambda'_{\perp})$. (Recall that the RG flows of λ^a and λ'^a are decoupled.) These one-loop renormalization-group flows are shown in Fig. 10. The separatrix $X^2 - Y^2 = 0$ is typical of the Kosterlitz-Thouless renormalization group flows. The RG flow diagram in Fig. 10 indicates that the fixed points for the interacting surface modes when $\lambda^a = \lambda'^a$ are (i) the $SU(2)$ -symmetric strong-coupling fixed point

$$\lambda^a = \lambda'^a \equiv \lambda \rightarrow \infty, \quad (5.7)$$

(ii) the strong-coupling fixed point

$$-\lambda^1 = -\lambda'^1 = -\lambda^2 = -\lambda'^2 = \lambda^3 = \lambda'^3 \equiv \lambda \rightarrow \infty, \quad (5.8)$$

that follows from performing a global $SU(2)$ rotation by $\pi/2$ about the quantization axis, and (iii) the line of fixed points

$$\lambda^1 = \lambda'^1 = \lambda^2 = \lambda'^2 = 0, \quad \lambda^3 = \lambda'^3 \equiv \lambda_{\parallel} < 0. \quad (5.9)$$

Case (i) [and, upon making a global $SU(2)$ rotation, case (ii)] is the $SU(2)$ -isotropic strong-coupling fixed point discussed earlier. Cases (i) and (ii) dominate the phase diagram in the sense that any initial conditions in three of the four quadrants of the X - Y plane will lead to one of those strong-coupling fixed points.

The line of stable fixed points in case (iii) constitutes what we call a “sliding parafermion liquid” (SPL). This set of fixed points is gapless—even if the wires are initially coupled with some finite values of λ_{\perp} and λ_{\parallel} , the wires decouple as the theory flows to the IR. (The use of the term “parafermion” refers to the fact that the decoupled chiral $su(2)_k$ CFTs contain parafermion degrees of freedom.) Thus, the SPL resembles the so-called “sliding Luttinger liquids,” which are another class of non-Fermi liquid in (2+1) dimensions [49–51, 55, 93]. These sliding phases have in common the fact that certain classes of perturbations are either irrelevant or marginally irrelevant and hence flow to zero in the IR. It would be interesting to investigate the SPL phase in more detail, but at present such a study is beyond the scope of this work.

In summary, we have shown that, for a variety of initial conditions on the couplings $\lambda^a = \lambda'^a$, the interactions (5.2a) lead to an $SU(2)$ -symmetric strong-coupling RG fixed point, even if the interaction itself breaks $SU(2)$ symmetry explicitly. The $SU(2)$ -broken fixed points (like the SPL) may constitute interesting strongly-correlated gapless phases (i.e., non-Fermi liquids).

B. Mean-field theory for $k = 2$

Having established the stability of the $SU(2)$ -symmetric strong-coupling fixed point, we now wish to investigate the nature of this fixed point. Specifically, we would like to know whether this fixed point is gapped or gapless when the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$ is not explicitly broken. This is equivalent to asking whether the phase transition between the two $\mathcal{T}_{\text{eff},\hat{y}}$ -conjugate gapped phases $\lambda > \lambda' \geq 0$ and $0 \leq \lambda < \lambda'$ is first- or second-order. Moreover, we would like to determine whether the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$ might be spontaneously broken by the interacting surface theory. For general k , the answers to these questions are difficult to determine. Rewriting the interaction (5.2a) at the $SU(2)$ -symmetric point (5.2b) in terms of the parafermion representation (3.4) recasts it as a correlated hopping process like (3.6). However, unlike in the $\mathcal{T}_{\text{eff},\hat{y}}$ -breaking case studied in Sec. III, the current-current interactions on neighboring bonds do not commute for general k , owing to the presence of the nonzero couplings $\lambda' = \lambda$. Furthermore, since not all $su(2)_k$ CFTs admit a free-field description, performing detailed calculations is intractable in general. (Although it may be possible to make progress using certain methods from the theory of integrable systems, like the thermodynamic Bethe ansatz, which has been used to study perturbed $su(2)_k$ and \mathbb{Z}_k CFTs [68, 94].)

However, the case $k = 2$ is special, for it is the simplest nontrivial example in which the $su(2)_k$ current algebra has a free-fermion representation. In the $k = 2$ case, we can rewrite the $su(2)_2$ current operators as [see also Eqs. (3.4) for $k = 2$]

$$\hat{J}_{M,y}^+ = \sqrt{2} \hat{\psi}_{M,y} \hat{\xi}_{M,y}, \quad (5.10a)$$

$$\hat{J}_{M,y}^- = \sqrt{2} \hat{\xi}_{M,y}^\dagger \hat{\psi}_{M,y} \quad (5.10b)$$

$$\hat{J}_{M,y}^3 = i \frac{1}{\sqrt{2}} \partial_M \hat{\phi}_{M,y}. \quad (5.10c)$$

Here, $\hat{\psi}_{M,y}$ is an M-moving real (Majorana) fermion operator with $M = L, R$ standing for left and right, respectively. They create and annihilate M-moving complex (Dirac) fermions, respectively. Moreover, they are related to the chiral boson operators $\hat{\phi}_{M,y}$ through the vertex operator

$$\hat{\xi}_{M,y} = : e^{+i\sqrt{1/2}\hat{\phi}_{M,y}} :, \quad (5.10d)$$

where $M = L, R$. Finally, the chiral currents $\hat{J}_{M,y}^3$ can be reexpressed in terms of the pair of Dirac fermion operators $\hat{\xi}_{M,y}^\dagger$ and $\hat{\xi}_{M,y}$ using the bosonization identity

$$\begin{aligned} \hat{\rho}_{M,y} &:= \hat{\xi}_{M,y}^\dagger \hat{\xi}_{M,y} \\ &\equiv -\frac{1}{2\pi\sqrt{2}} \partial_M \hat{\phi}_{M,y}, \end{aligned} \quad (5.10e)$$

where $M = L, R$ and $(-1)^R = -(-1)^L \equiv 1$, that defines the two chiral fermionic densities. Note that the wire-dependence of the chiral bosonic commutation relations (3.4h) was chosen so as to ensure that fermionic vertex operators in different wires anticommute at equal times. The same is true for the Majorana fermions, owing to Eq. (3.4e) with $k = 2$.

The representation (5.10) is a free-fermion representation because it can be used to rewrite the kinetic contribution (5.1) in terms of two decoupled sectors of noninteracting fermions, one for the real (Majorana) fermions and one for the complex (Dirac) fermions. This is to say that one can use fermionic coherent states to represent the partition function associated with the Hamiltonian (5.1) as a path integral over Grassmann variables with the free Lagrangian density [95]

$$\begin{aligned} \mathcal{L}_{0,x=0} &:= \sum_{y=0}^{L_y}{}' 2 (\psi_{L,y+1} i\partial_L \psi_{L,y+1} + \psi_{R,y} i\partial_R \psi_{R,y}) \\ &\quad + \sum_{y=0}^{L_y}{}' 2 (\xi_{L,y+1}^* i\partial_L \xi_{L,y+1} + \xi_{R,y}^* i\partial_R \xi_{R,y}). \end{aligned} \quad (5.11)$$

Here, we have set the velocities in the z -direction to unity. Passing to a Lagrangian formulation of the problem provides an enormous simplification relative to the case of general $k > 2$, where there is no such formulation.

With the free-fermion representation (5.10) in hand, we can now embark on a “traditional” mean-field analysis of the interacting problem in which neighboring wires are coupled via the current-current interactions. This will allow us to address the question of whether the surface theory is truly critical, as well as that of whether the interacting surface breaks spontaneously the symmetry $\mathcal{T}_{\text{eff},\hat{y}}$.

In the fermionic representation (5.11), the current-current interaction (5.2a) between nearest-neighbor wires on the surface takes the form

$$\begin{aligned} \mathcal{L}_{\text{bs},x=0} &= -\lambda \sum_{y=0}^{L_y}{}' \left[(\xi_{L,y+1}^* \xi_{R,y} \psi_{L,y+1} \psi_{R,y} + \xi_{R,y}^* \xi_{L,y+1} \psi_{R,y} \psi_{L,y+1}) - (2\pi)^2 \xi_{L,y+1}^* \xi_{R,y} \xi_{R,y}^* \xi_{L,y+1} \right] \\ &\quad + (\lambda \rightarrow \lambda', R \leftrightarrow L, y \rightarrow y+1), \end{aligned} \quad (5.12)$$

where we have set $\lambda_{\perp} = \lambda_{\parallel} \equiv \lambda$ and $\lambda'_{\perp} = \lambda'_{\parallel} \equiv \lambda'$ as we are considering the $SU(2)$ -symmetric limit. Next, we decouple this interaction with a Hubbard-Stratonovich transformation. That is to say, for each directed bond $\langle y, y+1 \rangle$, we introduce the complex-valued auxiliary field $\Delta_{\xi,y}(t, z)$, together with the real-valued auxiliary

field $\Delta_{\psi,y}(t, z)$. Similarly, for each directed bond $\langle y+1, y+2 \rangle$, we introduce the complex-valued auxiliary field $\Delta'_{\xi,y}(t, z)$, together with the real-valued auxiliary field $\Delta'_{\psi,y}(t, z)$. We then introduce the auxiliary Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{bs},x=0}^{\text{aux}} := & \frac{1}{\lambda} \sum_{y=0}^{L_y} [(\overline{\Delta}_{\xi,y} + \Delta_{\xi,y}) \Delta_{\psi,y} - (2\pi)^2 (\overline{\Delta}_{\xi,y} \Delta_{\xi,y} + \overline{\Delta}_{\xi,y} i \lambda \xi_{L,y+1}^* \xi_{R,y} - \Delta_{\xi,y} i \lambda \xi_{R,y}^* \xi_{L,y+1})] \\ & - \sum_{y=0}^{L_y} [(i \xi_{R,y}^* \xi_{L,y+1} - i \xi_{L,y+1}^* \xi_{R,y}) \Delta_{\psi,y} + (\overline{\Delta}_{\xi,y} + \Delta_{\xi,y}) i \psi_{L,y+1} \psi_{R,y}] \\ & + (\lambda \rightarrow \lambda', \Delta_{\xi,\psi} \rightarrow \Delta'_{\xi,\psi}, i \rightarrow -i, R \leftrightarrow L, y \rightarrow y+1). \end{aligned} \quad (5.13)$$

The auxiliary Lagrangian density (5.13) reduces to the original Lagrangian density (5.12) when the equations of motion for the auxiliary fields, namely,

$$\Delta_{\xi,y} = -i \lambda \xi_{L,y+1}^* \xi_{R,y}, \quad (5.14a)$$

$$\Delta_{\psi,y} = +i \lambda \psi_{L,y+1} \psi_{R,y}, \quad (5.14b)$$

and similarly for the primed fields, are imposed. Note that the phases of the complex auxiliary fields $\Delta_{\xi,y}$ and $\Delta'_{\xi,y}$ can be removed by a gauge transformation, e.g. $\xi_{R,y} \rightarrow e^{i\theta} \xi_{R,y}$.

Imposing the symmetry $\mathcal{T}_{\text{eff},\mathbf{y}}$ forces the constraints $\Delta_{\xi} = \Delta'_{\xi}$ and $\Delta_{\psi} = \Delta'_{\psi}$ among the Hubbard-Stratonovich fields. However, it is important to remember that, within a mean-field treatment of the theory

with the interaction (5.13), $\mathcal{T}_{\text{eff},\mathbf{y}}$ -symmetry can be broken spontaneously if Eqs. (5.14) develop vacuum expectation values that are not symmetric under $\Delta_{\xi,\psi} \leftrightarrow \Delta'_{\xi,\psi}$. Checking the self-consistency of such spontaneous-symmetry-breaking solutions is one of the primary goals of the present mean-field calculation.

At this point, the standard way to proceed is to integrate out both the Dirac fermions ξ_M and the Majorana fermions ψ_M and then to solve for the saddle point of the effective action involving only the Hubbard-Stratonovich fields. We first focus on the Majorana contribution to Eq. (5.13). Taking the continuum limit in the y -direction and linearizing around $k_y = \pi/2$ yields the full Euclidean action

$$S_{x=0}^{\text{aux mf } \psi} := \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \begin{pmatrix} c_{\mathbf{k}}^* & c_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} i\omega + v_{\xi} k_y & k_z - i m_{\xi} \\ k_z + i m_{\xi} & i\omega - v_{\xi} k_y \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}} \\ c_{-\mathbf{k}}^* \end{pmatrix}, \quad (5.15a)$$

where we have transformed to Fourier space along each wire and defined the velocity and mass

$$v_{\xi} := 2(\Delta_{\xi} + \Delta'_{\xi}), \quad m_{\xi} := 2(\Delta_{\xi} - \Delta'_{\xi}). \quad (5.15b)$$

The action $S_{x=0}^{\text{aux mf } \psi}$ is expressed in terms of complex fermions

$$c_{\mathbf{k}} := c_{k_y}(k_z) \quad (5.15c)$$

that are defined in terms of the Majorana fermions by

$$\psi_{R,y} = i(c_y^* - c_y), \quad \psi_{L,y+1} = c_{y+1}^* + c_{y+1}, \quad (5.15d)$$

for any directed bond $\langle y, y+1 \rangle$ with y even, and

$$\psi_{L,y+1} = c_y^* + c_y, \quad \psi_{R,y+2} = i(c_{y+1}^* - c_{y+1}), \quad (5.15e)$$

for any directed bond $\langle y+1, y+2 \rangle$ with y even, followed by taking the Fourier transform

$$c_y(k_z) := \frac{1}{\sqrt{\mathcal{N}}} \sum_{k_y} e^{+i k_y y} c_{k_y}(k_z). \quad (5.15f)$$

A similar treatment of the Dirac contribution to Eq. (5.13) yields the full Euclidean action

$$S_{x=0}^{\text{aux mf } \xi} := \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \begin{pmatrix} \xi_{R,k}^* & \xi_{L,k}^* \end{pmatrix} \begin{pmatrix} i\omega + k_z & +i v_\psi k_y + m_\psi \\ -i v_\psi k_y + m_\psi & i\omega - k_z \end{pmatrix} \begin{pmatrix} \xi_{R,k} \\ \xi_{L,k} \end{pmatrix}, \quad (5.16a)$$

where we have defined the velocity and mass

$$v_\psi := (\Delta_\psi + \Delta'_\psi) - \frac{(2\pi)^2}{2} v_\xi, \quad (5.16b)$$

$$m_\psi := (\Delta_\psi - \Delta'_\psi) - \frac{(2\pi)^2}{2} m_\xi, \quad (5.16c)$$

where v_ξ and m_ξ are defined in Eq. (5.15b).

At this point, it is already possible to see that the

surface theory

$$S_{x=0}^{\text{aux mf}} := S_{x=0}^{\text{aux mf } \xi} + S_{x=0}^{\text{aux mf } \psi} \quad (5.17)$$

for the case $k = 2$ will be gapless so long as $\mathcal{T}_{\text{eff}, \hat{y}}$ is not broken spontaneously. This is because the masses m_ξ and m_ψ vanish when $\Delta_\xi = \Delta'_\xi$ and $\Delta_\psi = \Delta'_\psi$ [see Eqs. (5.15b) and (5.16c)]. Thus, what remains to be checked is that, upon integration over the fermions, the saddle point of the resulting effective action has self-consistent solutions such that $\Delta_\xi = \Delta'_\xi$ and $\Delta_\psi = \Delta'_\psi$. Integrating out the real and complex fermions, we find the following set of four self-consistency equations for the masses and velocities:

$$m_\psi = \frac{4\lambda\lambda'}{\lambda + \lambda'} \int \frac{d^2k}{(2\pi)^2} \frac{m_\xi}{\sqrt{v_\xi^2 k_y^2 + k_z^2 + m_\xi^2}} - 2\pi^2 m_\xi + \frac{\lambda - \lambda'}{\lambda + \lambda'} (v_\psi + 2\pi^2 v_\xi), \quad (5.18a)$$

$$m_\xi = \frac{4\lambda\lambda'}{\lambda + \lambda'} \int \frac{d^2k}{(2\pi)^2} \frac{m_\psi}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} + \frac{\lambda - \lambda'}{\lambda + \lambda'} v_\xi, \quad (5.18b)$$

$$v_\psi = \frac{4\lambda\lambda'}{\lambda + \lambda'} \int \frac{d^2k}{(2\pi)^2} \frac{v_\xi k_y^2}{\sqrt{v_\xi^2 k_y^2 + k_z^2 + m_\xi^2}} - 2\pi^2 v_\xi + \frac{\lambda - \lambda'}{\lambda + \lambda'} (m_\psi + 2\pi^2 m_\xi), \quad (5.18c)$$

$$v_\xi = \frac{4\lambda\lambda'}{\lambda + \lambda'} \int \frac{d^2k}{(2\pi)^2} \frac{v_\psi k_y^2}{\sqrt{v_\psi^2 k_y^2 + k_z^2 + m_\psi^2}} + \frac{\lambda - \lambda'}{\lambda + \lambda'} m_\xi. \quad (5.18d)$$

Equations (5.18) constitute a set of four coupled self-consistency equations that must be solved simultaneously for the four unknowns m_ξ , m_ψ , v_ξ , and v_ψ . For general values of λ and λ' , this must be done numerically. We find that nontrivial solutions of Eqs. (5.18) exist, and that they exhibit the following general features. When $\lambda = \lambda'$, we find that $m_\xi = m_\psi = 0$ for all $\lambda > 0$. Thus, at the mean-field level, the surface of the $su(2)_2$ non-Abelian coupled-wire construction is a gapless liquid with both Dirac and Majorana degrees of freedom, so long as the symmetry $\mathcal{T}_{\text{eff}, \hat{y}}$ is not broken explicitly. When $\lambda \neq \lambda'$, we find solutions where the masses m_ξ and $m_\psi \neq 0$. This agrees with our earlier hypothesis that the surface develops a gap when the symmetry $\mathcal{T}_{\text{eff}, \hat{y}}$ is broken explicitly.

VI. CONCLUSIONS

In this paper, we have shown how to construct a family of (3+1)-dimensional non-Abelian topological phases. These phases inherit their non-Abelian character from the underlying $su(2)_k$ CFTs that describe the constituent interacting fermionic quantum wires in the decoupled limit. For the special case of $su(2)_2$, we showed explicitly how to determine the nature of the topological order by computing the ground-state degeneracy on the three-torus. This calculation relies on the operator algebra of the underlying CFTs that furnish the low-energy degrees of freedom for the coupled-wire construction, thus making explicit the connection between these CFTs and the emergent topological phase. We also examined the phase diagram of the surface for this family of topological phases, and showed explicitly for the case of $su(2)_2$ that they are gapless and protected by an analogue of TRS.

There are many open questions to be pursued in light of this work. First and foremost, it would be interesting to determine what (3+1)-dimensional topological quantum field theory describes the family of phases constructed here. In particular, one could ask whether and how these phases could be represented within the Crane-Yetter/Walker-Wang construction. Another avenue to pursue would be to try to construct these phases via parton constructions like the ones that have been carried out for some non-Abelian quantum Hall states [96, 97], and for the Abelian (3+1)-dimensional FTIs [43, 44]. Both of these approaches would build confidence that the family of $su(2)_k$ topological phases constructed in this work is truly a universality class of phases that can be realized in a variety of physical settings (and not just using coupled wires).

Another interesting avenue to pursue would be to investigate more deeply the nature of the $su(2)_k$ surface states for $k > 2$. While spontaneous breaking of the TRS analogue (and the concomitant opening of a gap on the surface) is always a possibility, it could be that these surface states constitute novel stable fractionalized non-Fermi liquid phases. The investigation of this class of models would likely need to rely on nonperturbative techniques, and could provide insights into conformal field theories in (2+1)-dimensional spacetime.

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Appendix A: The parafermion current algebra

We are going to review how the affine Lie algebra of level $k = 1, 2, 3, \dots$ for the compact connected Lie group $SU(2)$ can be represented in terms of parafermions as was done by Zamolodchikov and Fateev in Ref. [88].

1. Gaussian algebra

For any $\kappa > 0$, define the Euclidean action

$$S := \frac{\kappa}{2} \int d^2 \mathbf{x} (\partial \varphi)^2 \quad (\text{A1})$$

for the real-valued scalar field φ and the positive number $0 < \kappa \in \mathbb{R}$. Its two-point function is

$$\langle \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} \ln |\mathbf{x} - \mathbf{y}|^2 \quad (\text{A2})$$

up to an additive dimensionful constant that depends on the boundary condition imposed on the Laplacian. If we trade the complex coordinates $v \in \mathbb{C}$ and $w \in \mathbb{C}$ in two-dimensional Euclidean space for the Cartesian coordinates $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{y} \in \mathbb{R}^2$, respectively, then

$$|\mathbf{x} - \mathbf{y}|^2 = (v - w)(\bar{v} - \bar{w}) \quad (\text{A3})$$

and

$$\langle \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} [\log(v - w) + \log(\bar{v} - \bar{w})], \quad (\text{A4a})$$

$$\langle \partial_v \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{(v - w)}, \quad (\text{A4b})$$

$$\langle \partial_v \varphi(\mathbf{x}) \partial_w \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{(v - w)^2}, \quad (\text{A4c})$$

$$\langle \partial_{\bar{v}} \varphi(\mathbf{x}) \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{(\bar{v} - \bar{w})}, \quad (\text{A4d})$$

$$\langle \partial_{\bar{v}} \varphi(\mathbf{x}) \partial_{\bar{w}} \varphi(\mathbf{y}) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{(\bar{v} - \bar{w})^2}. \quad (\text{A4e})$$

There follows the chiral Abelian OPEs

$$\partial_v \varphi(\mathbf{x}) \varphi(\mathbf{y}) = -\frac{1}{4\pi\kappa} \frac{1}{(v - w)} + \dots, \quad (\text{A5a})$$

$$\partial_v \varphi(\mathbf{x}) \partial_w \varphi(\mathbf{y}) = -\frac{1}{4\pi\kappa} \frac{1}{(v - w)^2} + \dots, \quad (\text{A5b})$$

$$\partial_{\bar{v}} \varphi(\mathbf{x}) \varphi(\mathbf{y}) = -\frac{1}{4\pi\kappa} \frac{1}{(\bar{v} - \bar{w})} + \dots, \quad (\text{A5c})$$

$$\partial_{\bar{v}} \varphi(\mathbf{x}) \partial_{\bar{w}} \varphi(\mathbf{y}) = -\frac{1}{4\pi\kappa} \frac{1}{(\bar{v} - \bar{w})^2} + \dots, \quad (\text{A5d})$$

$$\partial_v \varphi(\mathbf{x}) \partial_{\bar{w}} \varphi(\mathbf{y}) = 0. \quad (\text{A5e})$$

The scaling dimension of the field $\partial_v \phi$ is

$$\Delta_{\partial_v \phi} = 1. \quad (\text{A6})$$

Another set of chiral Abelian OPEs follows from making the Ansatz

$$\phi(v, \bar{v}) =: \phi_L(v) + \phi_R(\bar{v}), \quad (\text{A7a})$$

$$\langle \partial_v \phi_L(v) \phi_L(w) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{v - w}, \quad (\text{A7b})$$

$$\langle \partial_v \phi_R(\bar{v}) \phi_R(\bar{w}) \rangle = -\frac{1}{4\pi\kappa} \frac{1}{\bar{v} - \bar{w}}, \quad (\text{A7c})$$

$$\langle \phi_R(v) \phi_L(\bar{w}) \rangle = 0, \quad (\text{A7d})$$

The holomorphic, ϕ_L , and antiholomorphic, ϕ_R , fields are uniquely defined up to the addition of holomorphic and antiholomorphic functions, respectively. One then deduces from

$$\langle e^{+ia\phi_L(v)} e^{-ia\phi_L(w)} \rangle = \frac{1}{(v-w)^{\frac{a^2}{4\pi\kappa}}}, \quad (\text{A8a})$$

$$\langle e^{+ia\phi_L(v)} e^{+ia\phi_L(w)} \rangle = 0, \quad (\text{A8b})$$

$$\langle e^{+ia\phi_R(\bar{v})} e^{-ia\phi_R(\bar{w})} \rangle = \frac{1}{(\bar{v}-\bar{w})^{\frac{a^2}{4\pi\kappa}}}, \quad (\text{A8c})$$

$$\langle e^{+ia\phi_R(\bar{v})} e^{+ia\phi_R(\bar{w})} \rangle = 0, \quad (\text{A8d})$$

$$\langle e^{\pm ia\phi_L(v)} e^{\pm ia\phi_R(\bar{w})} \rangle = 0, \quad (\text{A8e})$$

that

$$\begin{aligned} e^{+ia\phi_L(v)} e^{-ia\phi_L(w)} &= \frac{1}{(v-w)^{\frac{a^2}{4\pi\kappa}}} \\ &+ \frac{ia}{(v-w)^{\frac{a^2}{4\pi\kappa}-1}} (\partial_w \phi_L)(w) + \dots, \end{aligned} \quad (\text{A9a})$$

$$\begin{aligned} e^{+ia\phi_R(\bar{v})} e^{-ia\phi_R(\bar{w})} &= \frac{1}{(\bar{v}-\bar{w})^{\frac{a^2}{4\pi\kappa}}} \\ &+ \frac{ia}{(\bar{v}-\bar{w})^{\frac{a^2}{4\pi\kappa}-1}} (\partial_{\bar{w}} \phi_R)(\bar{w}) + \dots, \end{aligned} \quad (\text{A9b})$$

are the only chiral Abelian OPEs between the vertex fields $e^{\pm ia\phi_L(v)}$ and $e^{\pm ia\phi_R(\bar{v})}$ that are proportional to the identity operator to leading order.

At last, we shall need the OPEs

$$\partial_v \phi_L(v) e^{+ia\phi_L(w)} = -\frac{ia}{4\pi\kappa} \frac{1}{(v-w)} e^{+ia\phi_L(w)} + \dots, \quad (\text{A10a})$$

$$\partial_{\bar{v}} \phi_R(\bar{v}) e^{+ia\phi_R(\bar{w})} = -\frac{ia}{4\pi\kappa} \frac{1}{(\bar{v}-\bar{w})} e^{+ia\phi_R(\bar{w})} + \dots. \quad (\text{A10b})$$

In the following, we make the choice

$$\kappa = \frac{1}{8\pi}. \quad (\text{A11})$$

With this choice, the conformal weights of the vertex fields $\exp(ia\phi_L)$ and $\exp(ia\phi_R)$ are

$$(h_a, \bar{h}_a) \equiv (a^2, 0), \quad (h_{\bar{a}}, \bar{h}_{\bar{a}}) \equiv (0, a^2), \quad (\text{A12})$$

respectively. Moreover, the proportionality constant on the right-hand side of Eq. (A10) is $-2ai$.

2. Parafermion algebra

Let $k = 0, 1, 2, \dots$ be a positive integer. Define the holomorphic conformal weights

$$\Delta_l := \frac{l(k-l)}{k}, \quad l = 0, \dots, k-1. \quad (\text{A13a})$$

We posit the family of k local parafermion fields

$$I, \Psi_1(v), \dots, \Psi_{k-1}(v), \quad (\text{A13b})$$

where I is the identity operator with the scaling dimension $\Delta_0 \equiv 0$. For any $m, n = 0, \dots, k-1$, we impose the OPEs [88]

$$\Psi_m(v) \Psi_n(v') = \frac{C_{\Psi_m \Psi_n}^{\Psi_{m+n}} \Psi_{m+n}(v')}{(v-v')^{\Delta_m + \Delta_n - \Delta_{m+n}}} + \dots \quad (\text{A13c})$$

with the understanding that $m+n$ is defined modulo k , i.e.,

$$\Psi_0 \equiv \Psi_k \equiv I. \quad (\text{A13d})$$

The complex-valued number $C_{\Psi_m \Psi_n}^{\Psi_{m+n}}$ is called a structure constant. Demanding that the OPEs for the parafermions are associative fixes this structure constant to be the positive roots of [88]

$$\begin{aligned} \left(C_{\Psi_m \Psi_n}^{\Psi_{m+n}} \right)^2 &= \\ &\frac{\Gamma(m+n+1) \Gamma(k-m+1) \Gamma(k-n+1)}{\Gamma(m+1) \Gamma(n+1) \Gamma(k-m-n+1) \Gamma(k+1)}, \end{aligned} \quad (\text{A13e})$$

provided the normalization conditions

$$C_{\Psi_m \Psi_{k-m}}^{\Psi_k} = 1, \quad m = 0, \dots, k-1 \quad (\text{A13f})$$

are imposed.

An important consequence of (A13e) is the symmetry

$$C_{\Psi_m \Psi_n}^{\Psi_{m+n}} = C_{\Psi_n \Psi_m}^{\Psi_{m+n}} \quad m, n = 0, \dots, k-1, \quad (\text{A14})$$

under interchanging m and n . This is why

$$\Psi_n(v') \Psi_m(v) = (-1)^{\Delta_{m+n} - \Delta_m - \Delta_n} \Psi_m(v) \Psi_n(v'), \quad (\text{A15a})$$

where

$$\Delta_{m+n} - \Delta_m - \Delta_n = -\frac{2mn}{k} \equiv S_{m,n}^{(k)}. \quad (\text{A15b})$$

We shall call $\pi S_{m,n}^{(k)}$ the mutual (self) statistical angle between the parafermion m and the parafermion $n \neq m$ (when $n = m$).

Because the OPE between Ψ_m and Ψ_{k-m} gives the

identity operator, we shall use the notation

$$\Psi_m^\dagger \equiv \Psi_{k-m} \quad (\text{A16a})$$

for $m = 1, \dots, k-1$. The self statistical angle of the parafermion m is

$$S_{m,m}^{(k)} = -\frac{2m^2}{k}. \quad (\text{A16b})$$

The self statistical angle of the parafermion $k-m$ is

$$S_{k-m,k-m}^{(k)} = -\frac{2(k-m)^2}{k} = S_{m,m}^{(k)} \bmod \mathbb{Z}. \quad (\text{A16c})$$

The mutual statistics between parafermion m and $k-m$ is

$$S_{m,k-m}^{(k)} = -\frac{2m(k-m)}{k} = -S_{m,m}^{(k)} \bmod \mathbb{Z}. \quad (\text{A16d})$$

3. Parafermion representation of the $su(2)_k$ current algebra

The $su(2)_k$ current algebra is defined by the holomorphic current algebra [84]

$$J^a(v) J^b(w) = \frac{(k/2) \delta^{ab}}{(v-w)^2} + \frac{i\epsilon^{abc}}{(v-w)} J^c(w) + \dots \quad (\text{A17})$$

for any $a, b = 1, 2, 3$ together with its antiholomorphic copy. Without loss of generality, we consider only this holomorphic current algebra.

In the basis

$$J^\pm := J^1 \pm iJ^2, \quad J^3, \quad (\text{A18a})$$

the holomorphic current algebra (A17) reads

$$J^\pm(v) J^\pm(w) = 0 + \dots, \quad (\text{A18b})$$

$$J^+(v) J^-(w) = \frac{k}{(v-w)^2} + \frac{2}{(v-w)} J^3(w) + \dots, \quad (\text{A18c})$$

$$J^3(v) J^\pm(w) = \pm \frac{1}{(v-w)} J^\pm(w) + \dots, \quad (\text{A18d})$$

$$J^3(v) J^3(w) = \frac{(k/2)}{(v-w)^2} + \dots. \quad (\text{A18e})$$

We are going to verify that this current algebra can be represented in terms of the Gaussian boson ϕ from Sec. A1 and the pair of parafermions $\Psi_1 \equiv \Psi$ and $\Psi_{k-1} \equiv \Psi^\dagger$ from Sec. A2.

We make the Ansatz

$$\begin{aligned} J^+(v) &= \mathcal{N} \Psi_1(v) e^{+i\sqrt{\frac{1}{k}} \phi(v)} \\ &\equiv \mathcal{N} \Psi(v) e^{+i\sqrt{\frac{1}{k}} \phi(v)}, \end{aligned} \quad (\text{A19a})$$

$$\begin{aligned} J^-(v) &= \mathcal{N} e^{-i\sqrt{\frac{1}{k}} \phi(v)} \Psi_{k-1}(v) \\ &\equiv \mathcal{N} e^{-i\sqrt{\frac{1}{k}} \phi(v)} \Psi^\dagger(v), \end{aligned} \quad (\text{A19b})$$

$$J^3(v) = i\frac{\sqrt{k}}{2} (\partial_v \phi)(v), \quad (\text{A19c})$$

where we impose on $\partial_v \phi$ the Gaussian algebra

$$\partial_v \phi(v) \partial_w \phi(w) = -\frac{2}{(v-w)^2} + \dots, \quad (\text{A20a})$$

while we impose on Ψ and Ψ^\dagger the parafermion algebra

$$\Psi(v) \Psi(w) = \frac{C_{\Psi\Psi}^I}{(v-w)^{2(k-1)/k}} + \dots, \quad (\text{A20b})$$

$$\Psi^\dagger(v) \Psi^\dagger(w) = \frac{C_{\Psi^\dagger\Psi^\dagger}^I}{(v-w)^{2(k-1)/k}} + \dots, \quad (\text{A20c})$$

$$\Psi(v) \Psi^\dagger(w) = \frac{1}{(v-w)^{2(k-1)/k}} + \dots. \quad (\text{A20d})$$

The OPE (A18e) follows from the Ansatz (A19c) with the OPE (A20a). Because of the OPE (A10), we have the OPE

$$\partial_v \phi(v) e^{\pm i\sqrt{\frac{1}{k}} \phi(w)} = \mp i\sqrt{\frac{1}{k}} \frac{2}{(v-w)} e^{\pm i\sqrt{\frac{1}{k}} \phi(w)}. \quad (\text{A21})$$

The OPE (A18d) follows from the Ansatz (A19) with the OPE (A21). We thus see that the multiplicative factor $\sqrt{1/k}$ entering the argument of the vertex fields $\exp(\pm i\sqrt{1/k} \phi)$ is fixed by the condition that the two currents have the holomorphic conformal weight one. In turn, the normalization factor \mathcal{N} is fixed by the following considerations. Because of the OPEs (A13) and (A9), we have the OPE

$$\begin{aligned} J^+(v) J^-(w) &= \mathcal{N}^2 \Psi_1(v) \Psi_{k-1}(w) e^{+i\sqrt{\frac{1}{k}} \phi(v)} e^{-i\sqrt{\frac{1}{k}} \phi(w)} \\ &= \left(\frac{\mathcal{N}^2}{(v-w)^{1-\frac{1}{k}+1-\frac{1}{k}}} + \dots \right) \frac{1}{(v-w)^{\frac{2}{k}}} \left(1 + i\sqrt{\frac{1}{k}} (v-w)(\partial_w \phi)(w) + \dots \right) \\ &= \frac{\mathcal{N}^2}{(v-w)^2} + \frac{(2\mathcal{N}^2/k)}{(v-w)} J^3(w) + \dots. \end{aligned} \quad (\text{A22})$$

The leading singularity on the right-hand side of this OPE agrees with the one on the right-hand side of Eq. (A18c) if

$$\mathcal{N}^2 = k. \quad (\text{A23})$$

Finally, the vanishing OPE (A18b) follows from the fact that the OPE between any two vertex fields such that the \mathbb{C} -valued prefactors to the fields $\phi(v)$ and $\phi(w)$ in the arguments of the vertex fields are not of opposite sign, vanishes to leading order.

We close Sec. A3 by observing that the Ansatz (A19) is not unique. Indeed, the transformation

$$\Psi(v) \mapsto \Psi(v) e^{+i\alpha}, \quad (\text{A24a})$$

$$\Psi^\dagger(v) \mapsto \Psi^\dagger(v) e^{-i\alpha}, \quad (\text{A24b})$$

$$\phi(v) \mapsto \phi(v) - \sqrt{k} \alpha, \quad (\text{A24c})$$

leaves the $su(2)_k$ currents (A18a) invariant for any choice of the number α . The number α is defined modulo 2π and takes k inequivalent values $2\pi n/k$, $n = 0, \dots, k-1$.

Appendix B: The \mathbb{Z}_k conformal field theory

The parafermions defined in Appendix A represent currents in the \mathbb{Z}_k conformal field theory (CFT). The \mathbb{Z}_k CFT describes the long-wavelength properties of the critical point of a two-dimensional lattice model of classical \mathbb{Z}_k spins. It is characterized by the primary fields [84]

$$\Phi_n^m(v), \quad m = 0, \dots, k, \quad m+n = 0 \pmod{2}, \quad (\text{B1})$$

with $n \in \mathbb{Z}$. The integer n must be restricted to the range $(-m, m]$ using the relations

$$\Phi_n^m(v) \equiv \Phi_{n+2k}^m(v) \equiv \Phi_{n-k}^{k-m}(v). \quad (\text{B2})$$

Hence, the number of unique primary fields for a given k is restricted. The holomorphic conformal weight of the primary field $\Phi_n^m(v)$ is given by [84]

$$\Delta_n^m := \frac{m(m+2)}{4(k+2)} - \frac{n^2}{4k}, \quad (\text{B3})$$

which is nonnegative for $-m < n \leq m$. Among the primary fields $\Phi_n^m(v)$, there are the so-called “identity field”

$$\mathbb{1} := \Phi_0^0, \quad (\text{B4a})$$

the “twist fields”

$$\sigma_m := \Phi_m^m, \quad m = 1, \dots, k-1, \quad (\text{B4b})$$

and the parafermions

$$\Psi_m := \Phi_{2m}^0 \equiv \Phi_{2m-k}^k, \quad m = 1, \dots, k-1, \quad (\text{B4c})$$

which were introduced in Appendix A. The twist fields are of particular importance, as they are the continuum analogues of the lattice \mathbb{Z}_k spins.

The \mathbb{Z}_k primary fields obey the “fusion algebra” [84]

$$\Phi_n^m \times \Phi_{n'}^{m'} = \sum_{\substack{l=|m-m'| \\ l+m+m'=0 \pmod{2}}}^{\min(m+m', 2k-m-m')} \Phi_{n+n'}^l. \quad (\text{B5a})$$

This fusion algebra is a shorthand notation for the OPEs

$$\Phi_n^m(v) \Phi_{n'}^{m'}(w) = \sum_{\substack{l=|m-m'| \\ l+m+m'=0 \pmod{2}}}^{\min(m+m', 2k-m-m')} C_{\Phi_n^m \Phi_{n'}^{m'}}^{\Phi_{n+n'}^l} (v-w)^{S_{n \ n'}^{m \ m' \ l} (n+n')} \Phi_{n+n'}^l(v), \quad (\text{B5b})$$

where

$$S_{n \ n'}^{m \ m' \ l} (n+n') := \Delta_{n+n'}^l - \Delta_n^m - \Delta_{n'}^{m'}, \quad (\text{B5c})$$

and the structure constants $C_{\Phi_n^m \Phi_{n'}^{m'}}^{\Phi_{n+n'}^l}$ are fixed by associativity of the algebra. The quantity $2\pi S_{n \ n'}^{m \ m' \ l} (n+n')$ is the phase acquired, in the channel where m and m' fuse to l , when the complex coordinate v is rotated around the complex coordinate w .

1. Example: \mathbb{Z}_2 (Ising CFT)

When the conditions (B2) are imposed, the \mathbb{Z}_2 CFT (also known as the Ising CFT, as it describes the critical point of the classical Ising model in two dimensions) has the three primary fields

$$\mathbb{1}, \quad \sigma_1 \equiv \sigma, \quad \Psi_1 \equiv \psi. \quad (\text{B6})$$

According to Eq. (B3), their holomorphic conformal weights are

$$\Delta_{\mathbb{1}} = 0, \quad \Delta_{\sigma} := \frac{1}{16}, \quad \Delta_{\psi} = \frac{1}{2}, \quad (\text{B7})$$

respectively. According to Eq. (B5a), the primaries obey the fusion algebra

$$\sigma \times \sigma = \mathbb{1} + \psi, \quad (\text{B8a})$$

$$\psi \times \psi = \mathbb{1}, \quad (\text{B8b})$$

$$\sigma \times \psi = \sigma, \quad (\text{B8c})$$

in addition to the trivial fusion rules

$$\mathbb{1} \times a = a \quad (\text{B8d})$$

for $a = \mathbb{1}, \sigma, \psi$.

Appendix C: Commutation between string operators and the Hamiltonian; “Analytic” proof of state exclusion for the 2D case

1. Introduction

We are given the Hamiltonian

$$\hat{H}_{\text{bs}} := \int_0^{L_z} dz \hat{\mathcal{H}}_{\text{bs}} \quad (\text{C1})$$

and we are told that it commutes with two nonlocal operators $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$. Moreover, we are told that $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$ commute pairwise. Hence, we can label any eigenstate of the Hamiltonian \hat{H}_{bs} by the simultaneous eigenvalues ω_1^ψ and ω_2^ψ of the operators $\hat{\Gamma}_1^\psi$ and $\hat{\Gamma}_2^\psi$. In particular, we can label the basis for the ground-state manifold by

$$\{|\omega_1^\psi, \omega_2^\psi, \dots\rangle\} \quad (\text{C2})$$

where the \dots allow for additional sources of degeneracies. We shall demand that this basis is orthonormal.

In order to establish the set to which the eigenvalues ω_1^ψ and ω_2^ψ belong, we note that we are given two nonlocal operators

$$\hat{\Gamma}_1^\sigma(z) := \prod_{y=0}^{L_y} \hat{\sigma}_{\text{L},y}(z) \hat{\sigma}_{\text{R},y}(z), \quad (\text{C3a})$$

and

$$\begin{aligned} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) &:= \exp\left(\frac{i}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \hat{\phi}_{\text{R},y}(z)\right) \\ &\times \hat{\mathcal{P}}_1 \hat{\sigma}_{\text{R},y}(z_0) \hat{\sigma}_{\text{R},y}(z_0 + \epsilon) \hat{\mathcal{P}}_1 \\ &\equiv \hat{\mathcal{U}} \times \hat{\mathcal{P}}_1 \hat{\sigma}_{\text{R},y}(z_0) \hat{\sigma}_{\text{R},y}(z_0 + \epsilon) \hat{\mathcal{P}}_1. \end{aligned} \quad (\text{C3b})$$

The operator $\hat{\Gamma}_1^\sigma$ is a discrete product of a countable number of operators acting along a closed y -cycle of the two-torus. It requires no regularization for its definition, and it is nonunitary. It anticommutes with $\hat{\Gamma}_2^\psi$, and commutes

with $\hat{\Gamma}_1^\psi$ and with the Hamiltonian (C1). In contrast, the operator $\hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ is a nonlocal operator defined within one chiral channel of the wire y . It acts along an open string (with base point z_0 , along the z -cycle coinciding with wire y) that fails to close by the infinitesimal amount $\epsilon > 0$. It is nonunitary and it anticommutes with $\hat{\Gamma}_1^\psi$ in the limit $\epsilon \rightarrow 0$.

If both $\hat{\Gamma}_1^\sigma(z)$ and $\lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ were to commute with the Hamiltonian, then so would their product. The ground-state manifold would then be four-dimensional, with the orthogonal basis

$$|\Omega, \dots\rangle := |\omega_1^\psi, \omega_2^\psi, \dots\rangle, \quad (\text{C4a})$$

$$\hat{\Gamma}_1^\sigma(z) |\Omega, \dots\rangle \equiv \mathcal{N}_1 |\omega_1^\psi, -\omega_2^\psi, \dots\rangle, \quad (\text{C4b})$$

$$\lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) |\Omega, \dots\rangle \equiv \mathcal{N}_2 |-\omega_1^\psi, \omega_2^\psi, \dots\rangle, \quad (\text{C4c})$$

$$\hat{\Gamma}_1^\sigma(z) \left[\lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) |\Omega, \dots\rangle \right] \equiv \mathcal{N}_{12} |-\omega_1^\psi, -\omega_2^\psi, \dots\rangle, \quad (\text{C4d})$$

$$\dots \quad (\text{C4e})$$

We demand that the states on the left-hand side can be normalized. This can only be achieved if the normalizations \mathcal{N}_1 , \mathcal{N}_2 , and \mathcal{N}_{12} are neither zero nor infinity, for the basis (C2) is orthonormal by assumption. Here, we had to introduce three normalization factors to account for the fact that none of the operators $\hat{\Gamma}_1^\sigma(z)$, $\hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$, and $\hat{\Gamma}_1^\sigma(z) \times [\lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)]$ are nonunitary. However, the logical possibility that one or more of these normalizations are zero or infinity cannot be excluded. In this appendix, we will assume \mathcal{N}_1 and \mathcal{N}_2 to be non-vanishing and finite. This assumption amounts to choosing the “highest-weight state” (C4a) appropriately. The quantity \mathcal{N}_{12} could be determined by direct calculation, provided that the explicit form of the state $|\Omega, \dots\rangle$ is known. Since we do not have this knowledge, we leave its value unspecified for the moment.

Given that we do not know the value of \mathcal{N}_{12} , we proceed by an alternate route. This line of reasoning makes use of the fact that it is not correct to think of the operator $\lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ as commuting with the Hamiltonian (C1). It is a nonlocal operator that changes the topological sector of the state on which it acts, and can potentially exhibit different limiting behavior as a function of ϵ when acting on states belonging to different topological sectors. Thus, the limit $\epsilon \rightarrow 0$ must be treated carefully; the operators $\hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ and $\hat{\Gamma}_1^\sigma$ can interact highly nontrivially with one another in this limit, as we shall see. Indeed, instead of the set of states (C4), we can also consider the following set of states,

$$|\Omega, \dots\rangle := |\omega_1^\psi, \omega_2^\psi, \dots\rangle, \quad (\text{C5a})$$

$$|\hat{\Gamma}_1^\sigma, \dots\rangle := \hat{\Gamma}_1^\sigma(z) |\Omega, \dots\rangle, \quad (\text{C5b})$$

$$|\hat{\Gamma}_2^\sigma, \dots\rangle := \lim_{\epsilon \rightarrow 0} \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) |\Omega, \dots\rangle, \quad (\text{C5c})$$

$$|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma, \dots\rangle := \lim_{\epsilon \rightarrow 0} \left[\hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) |\Omega, \dots\rangle \right]. \quad (\text{C5d})$$

The only difference between the states (C5) and the states (C4) is that the limit $\epsilon \rightarrow 0$ is taken *after* forming the product $\hat{\Gamma}_1^\sigma \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ in Eq. (C5d). We adopt the point of view that the dimension of the ground-state manifold of the Hamiltonian (C1) cannot depend on the choice of when [i.e., before or after forming the product $\hat{\Gamma}_1^\sigma \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$] the limit $\epsilon \rightarrow 0$ is taken. Hence, the number of ground states present in Eqs. (C4) and (C5) must agree with one another. For this reason, we ask how many of the states (C5) are indeed ground states of the interaction (C1). This allows us to scrutinize the limiting behavior of operator products without losing important information related to the nonlocality of its constituent operators. We will show that the state (C5d) cannot be in the ground-state manifold of the interaction (C1). Logical consistency then demands that $\mathcal{N}_{12} = 0$ or ∞ in Eqs. (C4), as these are the only two possibilities that would exclude the state (C4d) from the ground-state manifold.

The nonunitary operator $\hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ does not commute with the interaction \hat{H}_{bs} defined by Eq. (C1). The purpose of this appendix is to determine whether the states (C5c) and (C5d), which involve taking the limit $\epsilon \rightarrow 0$, indeed belong to the ground-state manifold of the interaction (C1) once this limit is taken. More precisely, we define

$$\left[\hat{H}_{\text{bs}}, \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) \right] =: \hat{\mathcal{D}}_{2,y}(z_0, \epsilon), \quad (\text{C6a})$$

where the operator $\hat{\mathcal{D}}_{2,y}(z_0, \epsilon)$ is nonlocal, as we shall see below, and nonvanishing in general. We further define

$$\begin{aligned} \left[\hat{H}_{\text{bs}}, \hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) \right] &= \hat{\Gamma}_1^\sigma(z) \hat{\mathcal{D}}_{2,y}(z_0, \epsilon) \\ &=: \hat{\mathcal{D}}_{12,y}(z, z_0, \epsilon). \end{aligned} \quad (\text{C6b})$$

We are going to show that

$$\lim_{\epsilon \rightarrow 0} \hat{\mathcal{D}}_{2,y}(z_0, \epsilon) |\Omega, \dots\rangle = 0, \quad (\text{C7a})$$

Equation (C7a) is equivalent to the statement

$$\lim_{\epsilon \rightarrow 0} \left[\hat{H}_{\text{bs}}, \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) \right] |\Omega, \dots\rangle = (\hat{H}_{\text{bs}} - E_\Omega) |\hat{\Gamma}_2^\sigma, \dots\rangle = 0, \quad (\text{C7b})$$

where E_Ω is the energy eigenvalue of the state $|\Omega, \dots\rangle$. From this it immediately follows that the state $|\hat{\Gamma}_2^\sigma, \dots\rangle$ indeed belongs to the ground-state manifold of the interaction (C1).

We are also going to show that the state

$$\lim_{\epsilon \rightarrow 0} \hat{\mathcal{D}}_{12}(z, z_0, \epsilon) |\Omega, \dots\rangle \quad (\text{C8a})$$

has infinite norm as $z \rightarrow z_0$. Equation (C8a) is equivalent to the statement that

$$\begin{aligned} \lim_{z \rightarrow z_0} \lim_{\epsilon \rightarrow 0} \left[\hat{H}_{\text{bs}}, \hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon) \right] |\Omega, \dots\rangle \\ = \lim_{z \rightarrow z_0} \left(\hat{H}_{\text{bs}} - E_\Omega \right) |\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma, \dots\rangle \end{aligned} \quad (\text{C8b})$$

is a state with infinite norm. That this divergence occurs as $z \rightarrow z_0$ is especially problematic. In order for the product $\hat{\Gamma}_1^\sigma(z) \hat{\Gamma}_{2,y}^\sigma(z_0, \epsilon)$ of string operators to yield a topologically-degenerate ground state when acting on the state $|\Omega, \dots\rangle$, the resulting state cannot depend on the quantities z and z_0 in an observable way in the limit $\epsilon \rightarrow 0$. If this were the case, then the states $|\Omega, \dots\rangle$ and $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma, \dots\rangle$ could be distinguished by simply evaluating the string operator $\hat{\Gamma}_1^\sigma(z)$ near the point $z = z_0$. Hence, proving Eq. (C8a) will allow us to conclude that the state $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma, \dots\rangle$ does not belong to the ground-state manifold of the interaction (C1).

We are left with the conclusion of the paper, namely that the ground-state manifold of the interaction (C1) includes the states (C5a)–(C5c), and excludes the state (C5d). From now on, we ignore the \dots representing additional degeneracies for the ground-state manifold.

2. Calculation

We first prove Eq. (C7a). We begin by calculating $\hat{\mathcal{D}}_{2,y}(z_0, \epsilon)$, setting the base point $z_0 = 0$ without loss of generality. (Indeed, no physical quantity should depend on the choice of base point z_0 for a z -cycle.) For finite $\epsilon > 0$, we have

$$\hat{\mathcal{H}}_{\text{bs}} \hat{\Gamma}_{2,y}^\sigma(0, \epsilon) = \hat{\Gamma}_{2,y}^\sigma(0, \epsilon) \hat{\mathcal{H}}_{\text{bs}} \times \begin{cases} +1, & z > \epsilon, \\ +i, & z = \epsilon, \\ -1, & z < \epsilon. \end{cases} \quad (\text{C9})$$

We now use the definition (C6a), along with the identity

$$\hat{A} \hat{B} = \hat{B} \hat{A} f(z, \epsilon) \iff [\hat{A}, \hat{B}] = \hat{B} \hat{A} [f(z, \epsilon) - 1], \quad (\text{C10})$$

which gives

$$\hat{\mathcal{D}}_{2,y}(0, \epsilon) = -4i \int_0^\epsilon dz \sin \left(\frac{1}{\sqrt{2}} \left(\hat{\phi}_{\text{R},y}(z) - \hat{\phi}_{\text{L},y+1}(z) \right) \right) \hat{\mathcal{U}} \hat{\psi}_{\text{L},y+1}(z) \hat{\psi}_{\text{R},y}(z) \hat{\mathcal{P}}_1 \hat{\sigma}_{\text{R},y}(0) \hat{\sigma}_{\text{R},y}(\epsilon) \hat{\mathcal{P}}_1, \quad (\text{C11})$$

up to a contribution from the set of measure zero where $z = \epsilon$, which we will ignore.

To prove Eq. (C7a), we compute the leading contribution to $\widehat{\mathcal{D}}_2(\epsilon)$ as $\epsilon \rightarrow 0$. For ϵ infinitesimal, we may replace the integral in Eq. (C11) by the value of the integrand at the midpoint of the integration domain,

$$\widehat{\mathcal{D}}_{2,y}(0, \epsilon) \approx -4i\epsilon \sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \widehat{\mathcal{U}} \widehat{\psi}_{L,y+1}\left(\frac{\epsilon}{2}\right) \widehat{\psi}_{R,y}\left(\frac{\epsilon}{2}\right) \widehat{\mathcal{P}}_1 \widehat{\sigma}_{R,y}(0) \widehat{\sigma}_{R,y}(\epsilon) \widehat{\mathcal{P}}_1. \quad (\text{C12})$$

We now perform the (equal-time) OPE

$$\sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \widehat{\mathcal{U}} = \sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \exp\left(\frac{i}{2\sqrt{2}} \int_0^{L_z} dz \partial_z \widehat{\phi}_{R,y}(z)\right) \quad (\text{C13})$$

Inserting the OPEs

$$\lim_{\epsilon \rightarrow 0} e^{+\frac{i}{\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)} e^{-\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}(0)} \sim \frac{1}{\epsilon^{1/2}} e^{+\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)}, \quad (\text{C14a})$$

$$\lim_{\epsilon \rightarrow 0} e^{+\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}(L_z)} e^{-\frac{i}{\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)} \sim \frac{1}{\epsilon^{1/2}} e^{-\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)}, \quad (\text{C14b})$$

$$\lim_{\epsilon \rightarrow 0} e^{+\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}(L_z)} e^{+\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)} \sim \epsilon^{1/4} e^{+\frac{i}{\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)}, \quad (\text{C14c})$$

$$\lim_{\epsilon \rightarrow 0} e^{-\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)} e^{-\frac{i}{2\sqrt{2}} \widehat{\phi}_{R,y}(0)} \sim \epsilon^{1/4} e^{-\frac{i}{\sqrt{2}} \widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right)}, \quad (\text{C14d})$$

where “ \sim ” denotes equality up to constant factors and nonsingular terms, and using the fact that $L_z \sim 0$ by periodic boundary conditions, we find

$$\sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \widehat{\mathcal{U}} \sim \frac{1}{\epsilon^{1/4}} \sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right). \quad (\text{C15})$$

Next, we perform the OPE

$$\widehat{\mathcal{P}}_1 \widehat{\sigma}_{R,y}(0) \widehat{\sigma}_{R,y}(\epsilon) \widehat{\mathcal{P}}_1 \sim \frac{1}{\epsilon^{1/8}}. \quad (\text{C16})$$

Inserting this pair of OPEs into Eq. (C12), we find

$$\lim_{\epsilon \rightarrow 0} \widehat{\mathcal{D}}_{2,y}(0, \epsilon) |\Omega\rangle \sim \lim_{\epsilon \rightarrow 0} \epsilon^{5/8} \sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \widehat{\psi}_{L,y+1}\left(\frac{\epsilon}{2}\right) \widehat{\psi}_{R,y}\left(\frac{\epsilon}{2}\right) |\Omega\rangle = 0. \quad (\text{C17})$$

The form of the operator appearing on the RHS above is not important. All that matters is that its expectation value in the state $|\Omega\rangle$ is not singular in the limit $\epsilon \rightarrow 0$. Also of crucial importance is the factor $\epsilon^{5/8}$ that sends $\lim_{\epsilon \rightarrow 0} \widehat{\mathcal{D}}_{2,y}(0, \epsilon) |\Omega\rangle \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence, we may conclude that the state $|\widehat{\Gamma}_2^\sigma\rangle$, defined in Eq. (C5c), is a ground state.

We now turn to the state $|\widehat{\Gamma}_1^\sigma \widehat{\Gamma}_2^\sigma\rangle$, defined in Eq. (C5d), and ask if it, too, is a ground state. We will see that it cannot be a ground state by proving that the state defined in Eq. (C8a) has infinite norm as $z \rightarrow z_0$. We proceed by setting $z_0 = 0$, as before, and setting $z = z_0 = 0$ from the outset. Using Eq. (C12),

$$\begin{aligned} \widehat{\Gamma}_1^\sigma(0) \widehat{\mathcal{D}}_{2,y}(0, \epsilon) &\approx -4i\epsilon \left(\prod_{y'} \widehat{\sigma}_{L,y'}(0) \widehat{\sigma}_{R,y'}(0) \right) \sin\left(\frac{1}{\sqrt{2}} \left[\widehat{\phi}_{R,y}\left(\frac{\epsilon}{2}\right) - \widehat{\phi}_{L,y+1}\left(\frac{\epsilon}{2}\right)\right]\right) \widehat{\mathcal{U}} \\ &\times \widehat{\psi}_{L,y+1}\left(\frac{\epsilon}{2}\right) \widehat{\psi}_{R,y}\left(\frac{\epsilon}{2}\right) \widehat{\mathcal{P}}_1 \widehat{\sigma}_{R,y}(0) \widehat{\sigma}_{R,y}(\epsilon) \widehat{\mathcal{P}}_1 \end{aligned} \quad (\text{C18})$$

Using the OPEs (C13) in conjunction with the OPEs

$$\hat{\sigma}_{R,y}(0) \hat{\psi}_{R,y} \left(\frac{\epsilon}{2} \right) \sim \frac{1}{\epsilon^{1/2}} \hat{\sigma}_{R,y}(0) \quad (\text{C19a})$$

$$\hat{\sigma}_{L,y+1}(0) \hat{\psi}_{L,y+1} \left(\frac{\epsilon}{2} \right) \sim \frac{1}{\epsilon^{1/2}} \hat{\sigma}_{L,y+1}(0), \quad (\text{C19b})$$

we find

$$\hat{\Gamma}_1^\sigma(0) \hat{\mathcal{D}}_{2,y}(0, \epsilon) \sim \frac{1}{\epsilon^{3/8}} \sin \left(\frac{1}{\sqrt{2}} \left[\hat{\phi}_{R,y} \left(\frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left(\frac{\epsilon}{2} \right) \right] \right) \hat{\Gamma}_1^\sigma(0).$$

In contrast to the RHS of Eq. (C17), we now have the product between a local operator and a nonlocal operator on the RHS. Furthermore, the real-valued prefactor is a function of ϵ that diverges as $\epsilon \rightarrow 0$. We conclude that

$$\lim_{\epsilon \rightarrow 0} \hat{\mathcal{D}}_{12}(0, 0, \epsilon) |\Omega\rangle = \hat{\Gamma}_1^\sigma(0) \hat{\mathcal{D}}_{2,y}(0, \epsilon) |\Omega\rangle \sim \frac{1}{\epsilon^{3/8}} \sin \left(\frac{1}{\sqrt{2}} \left[\hat{\phi}_{R,y} \left(\frac{\epsilon}{2} \right) - \hat{\phi}_{L,y+1} \left(\frac{\epsilon}{2} \right) \right] \right) |\hat{\Gamma}_1^\sigma\rangle \quad (\text{C20})$$

is a state with infinite norm, as advertised, provided that the operator $\sin \left(\frac{1}{\sqrt{2}} [\hat{\phi}_{R,y}(\epsilon/2) - \hat{\phi}_{L,y+1}(\epsilon/2)] \right)$ does not annihilate the state $|\hat{\Gamma}_1^\sigma\rangle$. (Determining whether or not this is the case again requires an explicit expression for the state $|\Omega\rangle$, which we do not have at our disposal.) In that case, we conclude that the state $|\hat{\Gamma}_1^\sigma \hat{\Gamma}_2^\sigma\rangle$ cannot be a ground state of the interaction \hat{H}_{bs} defined by Eq. (C1).

Appendix D: Diagrammatics for operator algebra in the Ising CFT

The discussion surrounding Eqs. (3.18) in the main text concerns how to infer the exchange algebra of two chiral primary operators in the Ising CFT from their operator product expansion. This exchange algebra is simple to determine in cases where the two primary operators have a unique fusion product, as in the case of the σ and ψ operators in Eqs. (3.18). However, when the two primary operators *do not* have a unique fusion product, as occurs in the case of two σ operators [see the OPEs in Eqs. (3.19)], the exchange algebra depends on the fusion channel in which the product of the pair of operators is evaluated [see the exchange algebra in Eqs. (3.20)]. This poses a challenge for calculations: it is necessary to keep track of both fusion and braiding in a way that respects consistency conditions between the two. This challenge is the essence of the difference between Abelian and non-Abelian excitations in quantum field theory.

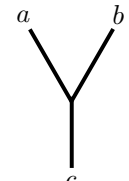
To this end, it is expedient to make use of the diagrammatic calculus developed in, e.g., Refs. [3, 5, 98, 99] to represent chiral algebras associated with rational conformal field theories (RCFTs). In this Appendix, we review aspects of this calculus, as they relate to the wire constructions of two- and three-dimensional non-Abelian topological phases discussed in this work. For simplicity, we focus on the example of the Ising CFT, although generalizations to other RCFTs are straightforward.

We first define the data necessary to compute the exchange algebra of chiral primary fields in a general RCFT. These are the fusion rules, the R -symbols, and the F -symbols. The fusion rules of the \mathbb{Z}_k RCFTs were given in Eq. (B5a), and for the special case of the Ising (\mathbb{Z}_2) RCFT in Eqs. (B8).

In general, for chiral primary fields a , b , and c , the fusion rules take the form

$$a \times b = \sum_c N_{ab}^c c, \quad (\text{D1a})$$

with N_{ab}^c nonnegative integers. The diagrammatic representation of a product of two chiral primary fields a and b that fuse to c is



$$\quad \quad \quad (\text{D1b})$$

The requirement that the fusion algebra (D1a) be associative imposes the constraints

$$\sum_d N_{ab}^d N_{dc}^e = \sum_f N_{af}^e N_{bc}^f. \quad (\text{D1c})$$

For many interesting RCFTs, including all of the \mathbb{Z}_k CFTs [c.f. Eq. (B5a)], the fusion coefficients $N_{ab}^c = 0$ or 1. For simplicity, we will restrict ourselves to this class of RCFTs, which is known as the class of RCFTs without fusion multiplicity since the nonnegative integers $N_{ab}^c = 0$ are never larger than one.

Read from bottom to top, diagram (D1b) is an element of the vector space V_c^{ab} , which is known as a “splitting space.” Read from top to bottom, it is an element of the vector space V_{ab}^c , which is known as a “fusion space.” These vector spaces are dual to one another, and we will

use the terms “fusion” and “splitting” interchangeably unless otherwise noted. The R -symbols are defined to be unitary maps

$$R_c^{ab} : V_c^{ba} \rightarrow V_c^{ab} \quad (\text{D2a})$$

that implement the diagrammatic braiding operation

$$\text{Diagram (D2b)} \quad (\text{D2b})$$

Note that we have defined the diagrammatic action of the R -symbols in such a way that the left leg of the fusion tree passes over the right leg. If instead the right leg passes over the left leg, then the inverse R -symbol $(R_c^{ab})^{-1}$ appears. The R -symbols are essential for determining how primary operators in an RCFT behave under exchange.

The final data necessary to determine the exchange algebra of primary operators in an RCFT are the F -symbols. These are required if exchange of chiral primary fields is to be associative. Associativity of the fusion rules (D1a) is encoded by Eq. (D1c). Equation (D1c) suggests that one defines the splitting space V_d^{abc} that encodes the fusion of three chiral fields a, b, c into one chiral field d by demanding that

$$\sum_e V_e^{ab} \otimes V_d^{ec} = \sum_f V_d^{af} \otimes V_f^{bc} \equiv V_d^{abc} \quad (\text{D3a})$$

holds. The F -symbols are then defined to be unitary maps

$$[F_d^{abc}]_{ef} : V_e^{ab} \otimes V_d^{ec} \rightarrow V_d^{af} \otimes V_f^{bc} \quad (\text{D3b})$$

that implement the diagrammatic operation

$$\text{Diagram (D3c)} \quad (\text{D3c})$$

The F -symbols F_d^{abc} are thus automorphisms (i.e., changes of basis) of the splitting space V_d^{abc} . The fusion rules, F -symbols, and R -symbols define a mathematical structure known as a braided fusion category (BFC). This structure can be used as a starting point for an axiomatic formulation of RCFT [5].

For the Ising RCFT, whose fusion rules are given in Eqs. (B8), the R -symbols are given by

$$R_1^{\sigma\sigma} = e^{+i\frac{\pi}{8}}, \quad (\text{D4a})$$

$$R_\psi^{\sigma\sigma} = e^{-i\frac{3\pi}{8}}, \quad (\text{D4b})$$

$$R_1^{\psi\psi} = -1, \quad (\text{D4c})$$

$$R_\sigma^{\psi\sigma} = R_\sigma^{\sigma\psi} = +i, \quad (\text{D4d})$$

with all other R -symbols trivial (i.e., equal to +1). Note that, up to complex conjugation, these R -symbols coincide with the phases acquired in Eqs. (3.18) and (3.20) when the corresponding chiral primary fields are exchanged. This is by design. The R -symbols reflect the monodromy of products of chiral primary fields in the corresponding RCFT. The F -symbols for the Ising RCFT are given by

$$F_\sigma^{\psi\psi\sigma} = F_\sigma^{\psi\sigma\psi} = F_\sigma^{\sigma\psi\psi} = -1, \quad (\text{D5a})$$

$$F_\psi^{\psi\sigma\sigma} = F_\psi^{\sigma\psi\sigma} = F_\psi^{\sigma\sigma\psi} = -1, \quad (\text{D5b})$$

$$F_\sigma^{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (\text{D5c})$$

with all other F -symbols trivial (i.e., equal to +1).

We will now demonstrate, using the example of the Ising RCFT, how to translate diagrams like those appearing in Eqs. (D2b) and (D3c) into algebraic statements. Performing this translation requires one to fix a chiral sector of the CFT. We choose to work with the chiral sector $M = R$. Once this choice is made, the starting point for this “dictionary” is to compare the diagram corresponding to the action of a particular R -symbol, say

$$\text{Diagram (D6a)} \quad (\text{D6a})$$

with its algebraic analogue, namely Eq. (3.18b),

$$\hat{\psi}_R(z) \hat{\sigma}_R(z') = \hat{\sigma}_R(z') \hat{\psi}_R(z) e^{+i\frac{\pi}{2} \text{sgn}(z-z')}, \quad (\text{D6b})$$

where we have suppressed the coordinate t as we assume all operators to be evaluated at equal times, and where we have suppressed the wire labels y, y' as we are working within a single chiral sector of a single CFT. Comparing Eqs. (D6a) and (D6b), we see that the phases only coincide if the diagram (D6a) is interpreted such that the coordinate z attached to the ψ branch is larger than the coordinate z' attached to the σ branch [i.e., if $\text{sgn}(z - z') = +1$]. We thus establish

Rule 1: In the operator product corresponding to a fusion tree, the spatial coordinates z , at which the operators are evaluated, are ordered according to the positions of the corresponding branches of the fusion tree on the axis pointing into the page. (D7)

As a sanity check of this rule, we note that if the ψ branch instead passed *over* the σ branch in the dia-

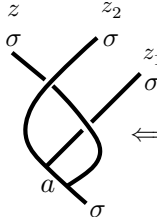
gram in Eq. (D6a), we would use $(R_\sigma^{\psi\sigma})^{-1} = -i$ instead, in accordance with Eq. (D2b), but the ordering of the legs would now dictate that $\text{sgn}(z - z') = -1$ in Eq. (D6b). Thus, Rule 1 ensures a meaningful correspondence between the R -symbols in the diagrammatics and the phases acquired under exchanging two operators in the CFT.

Next, we need to establish a convention for ordering the operators in an algebraic expression based on a fusion tree, and vice versa. There are various ways of doing this, but we choose to use

Rule 2: In the operator product corresponding to a fusion tree, the operators are ordered from *left to right* according to the order from *right to left* of the corresponding branches of the fusion tree, *before* any braiding is performed. (D8)

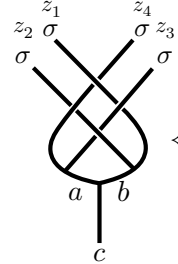
In Rule 2, the word “before” is interpreted under the assumption that the diagram is read from bottom to top. In this way, the ordering of operators in Eq. (D6b) agrees with the ordering of the branches of the fusion tree in Eq. (D6a).

With Rules 1 and 2 in place, we can now reliably translate fusion diagrams into equations and vice versa. For example, the correspondence



$$\hat{\sigma}_R(z) \hat{\sigma}_R(z_1) \hat{\sigma}_R(z_2) \quad (D9)$$

is used in Eqs. (3.66) and (3.67) of the main text, while the correspondence



$$\hat{\sigma}_R(z_1) \hat{\sigma}_R(z_2) \hat{\sigma}_R(z_3) \hat{\sigma}_R(z_4) \quad (D10)$$

is used in Eqs. (4.43) and (4.44).

Appendix E: Independence of string-operator algebra on arbitrary phase factors

We have made extensive use of the fact that the OPE of two operators in the *same* wire determines the algebra of these two operators under exchange. However, in certain situations [e.g. Eqs. (4.5) and (3.4)], we found it important (on physical grounds) to modify the exchange algebra between operators in *different* wires. We will now show that, despite their importance in calculating local quantities (such as the masses and velocities of the Majorana and Dirac fermions on the gapless surface of the 3D phase in Sec. VB), these modifications have no effect on topological features like the ground state degeneracy.

We proceed with an explicit example that illustrates how this comes about in the $(2+1)$ -dimensional $su(2)_2$ case studied in Sec. IIIC. We begin by rewriting the exchange algebra (3.18), but this time allowing for operators in different wires to have nontrivial commutation with one another. Hence, we posit that

$$\hat{\psi}_{M,y}(t, z) \hat{\sigma}_{M',y'}(t, z') = \hat{\sigma}_{M',y'}(t, z') \hat{\psi}_{M,y}(t, z) e^{+i \frac{\pi}{2} (-1)^M \delta_{M,M'} \delta_{y,y'} \text{sgn}(z-z')} e^{+i \epsilon_{M,M'} \delta_{y,y'} \varphi} e^{+i \text{sgn}(y-y') \theta_{M,M'}}, \quad (E1)$$

where $(-1)^R \equiv -(-1)^L \equiv 1$, $\epsilon_{R,L} = -\epsilon_{L,R} = 1$, and $\epsilon_{R,R} = \epsilon_{L,L} = 0$. The reason why the choice (E1) has no effect on the topological features of the phase is that all of these features depend on the algebra of string operators, which are constructed from bilinears in the operators $\hat{\psi}_{M,y}$ and $\hat{\sigma}_{M,y}$. In particular, for Majorana and twist-field operators in the same wire y , we have

$$\begin{aligned} \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z) \hat{\sigma}_{R,y}(t, z') \hat{\sigma}_{L,y}(t, z') &= \hat{\sigma}_{R,y}(t, z') \hat{\sigma}_{L,y}(t, z') \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z) \\ &\times e^{+i \epsilon_{L,R} \varphi} e^{+i \frac{\pi}{2} \text{sgn}(y-y')} e^{-i \frac{\pi}{2} \text{sgn}(y-y')} e^{+i \epsilon_{R,L} \varphi} \\ &= \hat{\sigma}_{R,y}(t, z') \hat{\sigma}_{L,y}(t, z') \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z). \end{aligned} \quad (E2)$$

For Majorana and twist-field operators in different wires $y \neq y'$, we find that

$$\begin{aligned} \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z) \hat{\sigma}_{R,y'}(t, z') \hat{\sigma}_{L,y'}(t, z') &= \hat{\sigma}_{R,y'}(t, z') \hat{\sigma}_{L,y'}(t, z') \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z) \\ &\times e^{+i \text{sgn}(y-y') (\theta_{L,R} + \theta_{L,L} + \theta_{R,R} + \theta_{R,L})} \\ &= \hat{\sigma}_{R,y'}(t, z') \hat{\sigma}_{L,y'}(t, z') \hat{\psi}_{R,y}(t, z) \hat{\psi}_{L,y}(t, z) \end{aligned} \quad (E3)$$

holds so long as the angles $\theta_{M,M'}$ satisfy

$$\theta_{L,R} + \theta_{L,L} + \theta_{R,R} + \theta_{R,L} \in 2\pi\mathbb{Z}. \quad (\text{E4a})$$

For general choices of the angles $\theta_{M,M'}$, Eq. (E4a) is automatically satisfied if

$$\theta_{R,R} = -\theta_{L,L}, \quad \theta_{L,R} = -\theta_{R,L}. \quad (\text{E4b})$$

Thus, when string operators are built from bilinears like

$\hat{\psi}_{R,y}\hat{\psi}_{L,y}$ and $\hat{\sigma}_{R,y}\hat{\sigma}_{L,y}$, the additional phases in the exchange algebra (E1) drop out of all calculations.

The calculations of the previous paragraph generalize readily to other combinations of primary operators, and to the three-dimensional case. The key observation in all cases is that string (and membrane) operators are built either from nonchiral bilinears of primary operators, like the ones studied in the previous paragraph, or from operators like $\hat{\mathcal{U}}_\alpha(t)$ [defined in Eq. (3.14)] that act only within one channel of one wire.

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